

Groundwater Hydrology - Homework 2

Due date: Nov. 26, 2012

1. In a confined, homogeneous and isotropic aquifer the following steady piezometric heads have been measured in three different wells A, B and C:

well	A	B	C
East coord. (m)	0	300	0
North coord.(m)	0	0	200
head (m)	+10	+11.5	+8.4

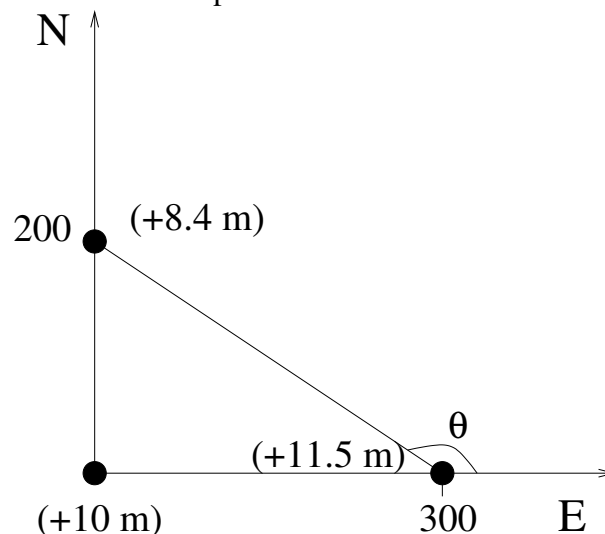
The aquifer is characterized by a thickness and a hydraulic conductivity equal to 20 m and 15 m/day, respectively.

Assuming a planar piezometric surface, determine:

- direction and magnitude of the hydraulic gradient I ;
- total rate that flows through the aquifer (per unit length in the direction orthogonal to the maximum gradient);
- Darcy's velocity at point P=(100 m, 100 m).

Solution

- a) The scenario is shown in the next plot:



Since the piezometric surface is a plane, thus with a constant slope, we calculate the components of the gradient along East and North from the well couples A-B and A-C to obtain:

$$I_{Est} = \frac{\partial h}{\partial x} = \frac{11.5 - 10}{300} = 5 \times 10^{-3}$$

$$I_{Nord} = \frac{\partial h}{\partial y} = \frac{8.4 - 10}{200} = -8 \times 10^{-3}$$

Hence $I = \sqrt{I_{Est}^2 + I_{Nord}^2} = 9.43 \times 10^{-3}$ Direction θ is given by:

$$\theta = \arctan\left(\frac{I_{Est}}{I_{Nord}}\right) = 122^\circ = 2.13rad$$

b) Total flow rate is:

$$Q = k \times b \times I \times 1 = 2.8 \text{ m}^3/\text{day}$$

c) Darcy's velocity is:

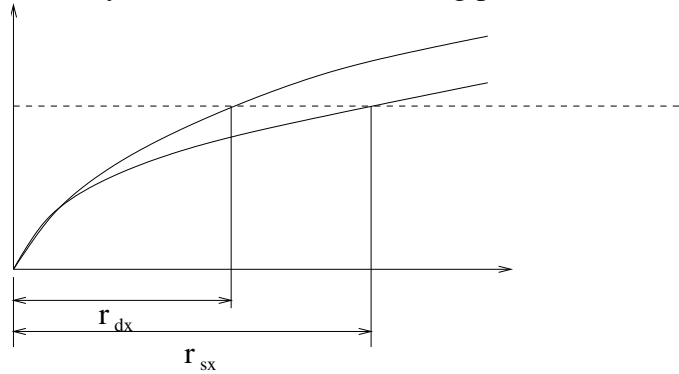
$$v = k \times I = 15 \times 9.43 \times 10^{-3} = 0.141 \text{ m/day}$$

2. Figure 1(a) represents maps of the contour lines of piezometric head drawdown at a given time t^* . They are related to water extraction from a well at a given flow rate Q in two different aquifers having the same transmissivity. Which of the two aquifers is characterized by a larger storativity and why?

Figure 1(b) represents maps of the contour lines of piezometric head as measured in two confined aquifers under steady state conditions. Evaluate the ratio between the incoming water flux from the left boundary in the two different cases, assuming that the aquifers have the same hydrogeological characteristics.

Solution

Fig (a): The isoline -2 m in the left figure is farther away from the well with respect to the corresponding isoline in the right figure. Note that, with the hypothesis of equal time, hydraulic conductivity and flow rate, the head gradient at the well must be also equal in the two configurations. Qualitatively we can draw the following plot:



We can see that the transient is slower in the right plot, (higher curve) which is thus characterized by a higher storage coefficient.

In other words, since a higher storage coefficient is related to a larger drainable water volume (per unit time), for an equal flow rate the pressure decline will propagate slower in the right case.

Fig (b): The contour line -10 m is located at a distance of 1 m and 3 m from the left boundary in the left and right figures, respectively. The incoming fluxes can be estimated as:

$$q_{left} = K \frac{10}{1} \quad q_{right} = K \frac{10}{3}$$

Hence the flux of the left figure is three times larger with respect to the case depicted in the right figure.

3. Figure 2 shows a contour representation of the piezometric surface for a confined aquifer discharging into a river. Measurements of hydraulic heads made at 17 monitoring wells are shown in the following Table:

Monitoring Well	Piezometric Head (m asml)	Monitoring Well	Piezometric Head (m asml)	Monitoring Well	Piezometric Head (m asml)
1	305.21	7	308.56	13	309.82
2	302.91	8	304.50	14	311.52
3	303.13	9	308.56	15	306.30
4	303.90	10	308.21	16	307.18
5	305.13	11	307.21	17	303.30
6	306.90	12	310.52		

The aquifer has an average thickness of 16 m and is considered as homogeneous and isotropic, with a porosity of 0.29 and a hydraulic conductivity of 4.3×10^{-4} m/s. Assuming steady-state conditions, estimate:

- the average hydraulic gradient in the aquifer;
- the rate of water discharge from the aquifer to the stream;
- the times it would take to a hypothetical tracer injected in Well # 16 to reach the river.

Solution

- There are many ways to solve this problem. One is to find an average gradient by fitting a plane through the data points by means of least square methods. This can be done by means of the following developments.

Given N observations $x_i, y_i, z_i, i = 1, \dots, N$, we try to approximate these data with a plane. The equation for a plane (linear function in 3D) is:

$$f(x, y) = z = ax + by + c$$

We define the sum of squares as

$$S(a, b, c) = \sum_{i=1}^N [ax_i + by_i + c - z_i]^2$$

The determination of the coefficients a, b, c proceeds by minimizing this function with respect to these coefficients, i.e., by setting the partial derivatives to zero as follows:

$$\begin{aligned} \frac{\partial S}{\partial a} &= 0 = 2 \sum_{i=1}^N [ax_i + by_i + c - z_i] x_i \\ \frac{\partial S}{\partial b} &= 0 = 2 \sum_{i=1}^N [ax_i + by_i + c - z_i] y_i \\ \frac{\partial S}{\partial c} &= 0 = 2 \sum_{i=1}^N [ax_i + by_i + c - z_i] \end{aligned}$$

This leads to the following linear system:

$$\begin{bmatrix} \sum_{i=1}^N x_i^2 & \sum_{i=1}^N x_i y_i & \sum_{i=1}^N x_i \\ \sum_{i=1}^N x_i y_i & \sum_{i=1}^N y_i^2 & \sum_{i=1}^N y_i \\ \sum_{i=1}^N x_i & \sum_{i=1}^N y_i & N \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^N x_i z_i \\ \sum_{i=1}^N y_i z_i \\ \sum_{i=1}^N z_i \end{bmatrix}$$

The following Matlab (Octave) code can be used for this:

```
#! /usr/bin/octave -qf

h_coord=load('hw2-ex3.dat');
x=h_coord(:,1);
y=h_coord(:,2);
h=h_coord(:,3);

one=ones(n,1);

a11=dot(x,x);
a22=dot(y,y);
a33=n;
a12=dot(x,y);
a21=a12;
a13=dot(x,one);
a31=a13;
a23=dot(y,one);
a32=a23;

b1=dot(x,h);
b2=dot(y,h);
b3=dot(h,one);

A=[ a11, a12, a13; a21, a22, a23; a31, a32, a33]

b=[ b1; b2; b3]

A\b
```

Where the file hw2-ex3.dat is:

1026.99	878.496666667	305.21
1139.06466667	1454.15133333	302.91
1620.36266667	1820.54333333	303.13
2626.598	1775.092	303.90
1816.60533333	1494.906	305.13
1536.41933333	1168.87133333	306.90
2356.60066667	1061.89066667	308.56
2300.56333333	1601.886	304.50

1755.474	791.893333333	308.56
1403.968	679.819333333	308.21
848.69	542.273333333	307.21
2224.14866667	476.047333333	310.52
2692.82333333	796.988	309.82
2458.486	633.970666667	311.52
2718.29533333	1377.73733333	306.3
2061.13133333	1230.00266667	307.18
2260.56333333	1945.092	303.30

The results from running the code is:

ans =

```

2.0986e-03
-5.3675e-03
3.0881e+02

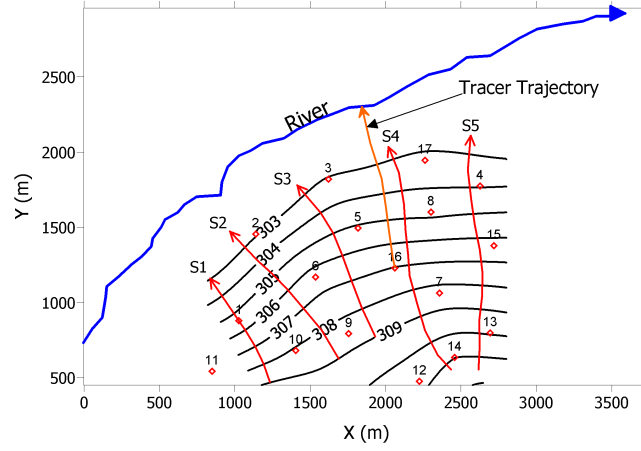
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which gives an average gradient:

$$i = \sqrt{(-5.3675 \times 10^{-3})^2 + (2.0986 \times 10^{-3})^2} = 5.763 \times 10^{-3}$$

Another way is to fit $\frac{\partial h}{\partial x}$ $\frac{\partial h}{\partial y}$ as 1D linear functions using least squares (this is the less accurate, obviously, and if the data are clustered in some strange way (not this case) we can get large errors).

Another way is to set up streamlines, as shown in the figure:



and calculate gradients with that. We obtain the following table:

Streamline	l (m)	Δh (m)	$\Delta h/l$
1	850	-6	-7.06×10^{-3}
2	1050	-6	-5.71×10^{-3}
3	1100	-6	-5.45×10^{-3}
4	1500	-8	-5.33×10^{-3}
5	1400	-8	-5.71×10^{-3}
Average gradient $i = 5.86 \times 10^{-3}$			

In this case, all the methods give similar results.

(b) From the last technique we obtain:

$$q = -b \cdot K \cdot i_{16} = 4.3 \times 10^{-4} \cdot 5.86 \times 10^{-3} = 4 \times 10^{-3} \frac{\text{m}^3}{\text{s m}} = 3.48 \frac{\text{m}^3}{\text{day m}}$$

We could have used the gradient as calculated from the 303 contour using again streamlines, and we would have obtained a slightly larger q .

(c) Along the streamline, the average gradient is $i \approx (307 - 302)/1100 = 0.0045$. Then we have:

$$v_s = -\frac{K}{\omega} i = 6.74 \times 10^{-6} \text{ m/s}$$

Thus the time for the particle to travel from well 16 to the stream is

$$t = \frac{d}{v_s} = 1100/6.74 \times 10^{-6} = 1.63 \times 10^8 \text{ s} = 5.2 \text{ years}$$

4. Given the unconfined aquifer of Figure 3.

- (a) Calculate the flow rate Q_w in the well of Figure 3 (left) assuming steady state flow and radial symmetry.
- (b) Under the same assumptions, calculate the flow rate Q_w in the well in Figure 3 (right).

Solution

- (a) Assuming Dupuit-type flow, we can write the flow equation of the aquifer in steady state conditions and homogeneous aquifer as:

$$\frac{1}{r} \frac{d}{dr} \left(r h \frac{dh}{dr} \right) = \frac{1}{r} \frac{d}{dr} \left(\frac{1}{2} r \frac{dh^2}{dr} \right) = 0$$

with boundary conditions:

$$\begin{aligned} Q &= \pi r_w K \frac{d(h^2)}{dr} && \text{at the well } (r = r_w) \\ h &= H && r = r_e \end{aligned}$$

Integrating this equation once we obtain:

$$\frac{r}{2} \frac{dh^2}{dr} = a$$

i.e.:

$$d(h^2) = a \frac{dr}{r}$$

or:

$$h^2(r) = a \ln r + b$$

From the first boundary condition, for $r = r_w$:

$$Q = \pi K a \quad \Rightarrow \quad a = \frac{Q}{\pi K}$$

From the second conditions, for $r = r_e$:

$$H^2 = \frac{Q}{\pi K} \ln r_e + b \quad \Rightarrow \quad b = H^2 - \frac{Q}{\pi K} \ln r_e$$

The solution of the problem is then:

$$h^2(r) - H^2 = \frac{Q}{\pi K} \ln \frac{r}{r_e}$$

Thus we can write:

$$Q = \pi K (h^2(r_w) - H^2) / \left(\ln \frac{r_w}{r_e} \right) = 4748 \text{m}^3/\text{day}$$

(b) The heterogeneities are parallel to the flow, thus we use weighted arithmetic mean:

$$\bar{K} = \frac{30 \cdot 40 + 15 \cdot 20}{45} = 33.33 \text{ m/day}$$

Thus we have:

$$Q = \frac{4748 \cdot 33.33}{40} = 3956.3 \text{ m}^3/\text{day}$$

A more accurate solution would be to split up the aquifer in two parts: the lower 15 m can be considered as a confined aquifer, the upper 30 m as an unconfined aquifer. Correspondingly we have:

$$Q_w = 2\pi r \left[K_2 a \frac{\partial h}{\partial r} + K_1 (h - a) \frac{\partial h}{\partial r} \right]$$

Integrating between $r = r_w$ and $r = r_e$ and between $h = h_w$ and $h = H$ we obtain: ($r_w = 0.3 \text{ m}$, $r_e = 300 \text{ m}$, $h_w = 42 \text{ m}$, $H = 75 \text{ m}$):

$$Q_w \int_{r_w}^{r_e} \frac{1}{r} dr = 2\pi \left[K_2 a \int_{h_w}^H dh + K_1 \int_{h_w}^H (h - a) dh \right]$$

from which:

$$\begin{aligned} Q_w &= \frac{2\pi}{\ln \frac{r_e}{r_w}} \left\{ K_2 a (H - h_w) + \frac{K_1}{2} [(H - a)^2 - (h_w - a)^2] \right\} \\ &= \frac{2\pi}{\ln 1000} \left[20 \cdot 15 \cdot (3) + \frac{40}{2} (30^2 - 27^2) \right] \\ &= 3929 \text{ m}^3/\text{day} \end{aligned}$$

As we can see the differences between the two methods are very small and are due to the fact that the mean conductivity does not take into account the changes in head (and thus transmissivity Kh) close to the well.

5. Given the unconfined aquifer of Figure 4, assuming a uniform recharge flux e , calculate:

- the mathematical model governing this problem assuming that Dupuit assumption holds and that the hydrogeological characteristics of the aquifer are constant (include initial and boundary conditions);
- the function $h(x)$ describing the water table in steady state conditions. Assume: $H_1 = 20$ m, $H_2 = 10$ m, $K = 5 \times 10^{-3}$ m/s, recharge $e = 7.2$ mm/h, aquifer length $L = L_1 + L_2 = 2000$ m.
- A ditch is to be excavated (along the y axis) in the mid section of the aquifer. What is the maximum depth of the ditch so that no pumping is required for its excavation. (Make appropriate assumptions if necessary.)

Solution

- Using Dupuit assumption the 1-D steady state equation of water flow in an unconfined aquifer subject to a recharge flux e is:

$$\frac{\partial}{\partial x} \left(Kh \frac{\partial h}{\partial x} \right) = e$$

Integration of this equation leads to:

$$\begin{aligned} Kh \frac{\partial h}{\partial x} &= -ex + c'_1 \\ \frac{\partial h^2}{\partial x} &= \frac{e}{2K}x + c_1 \\ h^2(x) &= \frac{e}{K}x^2 + c_1x + c_2 \end{aligned}$$

- Using the boundary conditions $h(0) = H_1$ and $h(L) = H_2$, with $L = L_1 + L_2$, we obtain:

$$h(x) = \sqrt{\frac{e}{K}(x^2 - Lx) + \frac{H_2^2 - H_1^2}{L}x + H_1^2}$$

where the convention is that a negative e corresponds to recharge. In our case we have:

$$\begin{aligned} h(x) &= \sqrt{\frac{7.2 \cdot 10^{-3}}{5 \times 10^{-3} \cdot 3600}(x^2 - 2000x) + \frac{10^2 - 20^2}{2000}x + 20^2} \\ h(x) &= \sqrt{-0.0004(x^2 - 2000x) - 0.15x + 400} \end{aligned}$$

- calculate the value of x at which the water table has a zero slope:

$$\frac{1}{2}h \frac{\partial h}{\partial x} = \frac{\partial h^2}{\partial x} = 0 = -0.0008x + 0.65$$

that yields: $x^* = 812.5$ m. At that distance we have the maximum height of our water table, which we can set equal to the maximum excavation elevation required by the question:

$$h(812.5) = \sqrt{-0.0004(812.5^2 - 2000 \cdot 812.5) - 0.15 \cdot 812.5 + 400} = 25.77 \text{ m}$$

The maximum excavation depth is thus $30 - 25.77 = 4.23$ m

6. Given the confined aquifer of Figure 5, determine the flow rate (per unit width in the direction orthogonal to the drawing) and draw an accurate plot of the behavior of the piezometric head between the two wells.

Solution The flowrate per unit width can be calculated using an average conductivity: since the heterogeneities are orthogonal to the flow, we need to use the harmonic mean:

$$Q = BK_M \frac{\Delta h}{L}$$

where ($L = \sum L_i$):

$$\begin{aligned} \frac{L}{K_M} &= \frac{L_1}{K_1} + \frac{L_2}{K_2} + \frac{L_3}{K_3} \\ &= \frac{200}{40} + \frac{800}{10} + \frac{300}{20} = 13 \text{m/day} \end{aligned}$$

and thus:

$$Q = 50 * 13 * \frac{4}{1300} = 2 \frac{\text{m}^3}{\text{day m}}$$

The flowrate is constant in every section, so we have:

$$Q = T_i \frac{\Delta H_i}{L_i} \quad i = 1, 2, 3$$

from which:

$$\begin{aligned} \Delta H_1 &= \frac{QL_1}{T_1} = \frac{2 * 200}{50 * 40} = 0.2 \text{m} \\ \Delta H_2 &= \frac{QL_2}{T_2} = \frac{2 * 800}{50 * 10} = 3.2 \text{m} \\ \Delta H_3 &= \frac{QL_3}{T_3} = \frac{2 * 300}{50 * 20} = 0.6 \text{m} \end{aligned}$$

7. The aquifer system in Figure 6 is formed by two confined aquifers characterized by transmissivities T_1 and T_2 , respectively. Water enters in the upper right portion of the system and exits as a spring in the lower left. Determine the head in the observation well M and calculate the conditions under which well M is artesian (water flows naturally out of the well).

Solution

- (a) This aquifer, being confined, can be treated as a planar aquifer with lengths equal to $l_1 = L_1 / \cos \beta$ and $l_2 = L_2 / \cos \alpha$. Then we can define two linear solutions in the two subdomains and impose the compatibility conditions (we put the zero of the x -axis at point M):

$$\begin{aligned} h_1(x) &= a_1x + b_1 \\ h_2(x) &= a_2x + b_2 \\ h_1(-l_1) &= H_1 \\ h_2(l_2) &= H_2 \\ h_1(0) &= h_2(0) (= H_M \text{ which is unknown}) \\ T_1 \frac{dh_1}{dx} \Big|_{x=0} &= T_2 \frac{dh_2}{dx} \Big|_{x=0} \end{aligned}$$

From the last equations, and noting that for a linear function the gradients are constant and can be calculated using H_1 , H_M , and H_2 , we can write:

$$T_1 \frac{dh_1}{dx} = T_1 \frac{H_M - H_1}{l_1} = T_2 \frac{dh_2}{dx} = T_2 \frac{H_2 - H_M}{l_2}$$

from which we obtain:

$$H_m = \frac{T_1 l_2 H_1 + T_2 l_1 H_2}{T_1 l_2 + T_2 l_1}$$

Now, since $L_1 = l_1 \cos \beta$ and $L_2 = l_2 \cos \alpha$, we also have that $H_1 = L_1 \tan \beta$ and $H_2 = L_2 \tan \alpha$. Thus we can write:

$$H_m = \frac{T_1 \sin \beta + T_2 \sin \alpha}{\frac{T_1 \cos \beta}{L_1} + \frac{T_2 \cos \alpha}{L_2}}$$

- (b) The top of well shown in the Figure (point M) is at an height given by the linear interpolation between H_1 and H_2 , i.e.,

$$H_{\text{well}} = \frac{H_2 - H_1}{L_1 + L_2} L_1 + H_1$$

To calculate the conditions for the aquifer for beeing artesian, we impose that $H_M > H_{\text{well}}$. The limiting condition is when:

$$\frac{L_2 H_1 + L_1 H_2}{L_1 + L_2} = \frac{T_1 l_2 H_1 + T_2 l_1 H_2}{T_1 l_2 + T_2 l_1}$$

which is verified for example when:

$$\begin{aligned} T_2 l_1 &= L_1 \\ T_1 l_2 &= L_2 \end{aligned}$$

8. Figure 7 reports two different configurations of wells and pizeometer in a confined aquifer, characterized by a hydraulic conductivity $K = 10$ m/day, an elastic storage coefficient $S_s = 10^{-5}$ m⁻¹ and a thickness $B=10$ m. From the wells, water is extracted at the following rates: $Q_1 = 15$ m³/day for 50 days; $Q_2 = 10$ m³/day for 20 days; $Q_2 = 20$ m³/day for the next 30 days; $Q_3 = 20$ m³/day for 50 days. Calculate the drawdown in the piezometer W at $t = 50$ days for the two distinct configurations (a) and (b).

Solution

$$T = 100\text{m}^2/\text{day} \quad S = 10^{-4}$$

a)

$$u_{p1} = \frac{4Tt}{r^2S} = \frac{4 \cdot 100 \cdot 50}{100^2 \cdot 10^{-4}} = 0.2 \times 10^5 > 100 \quad \text{apply Cooper-Jacob}$$

$$\Delta h_{p1} = \frac{Q_1}{4\pi T} \ln \left(2.25 \frac{Tt}{r^2S} \right) = 0.111 \text{ m}$$

$$u_{p2} = \frac{4Tt}{r^2S} = \frac{4 \cdot 100 \cdot 20}{70^2 \cdot 10^{-4}} = 0.167 \times 10^5 > 100 \quad \text{apply Cooper-Jacob}$$

$$\Delta h_{p2,50\text{days}} = \Delta h_{Q_{2A},50\text{days}} + \Delta h_{Q_{2B},30\text{days}}; \quad Q_{2A} = 10\text{m}^3/\text{day}; Q_{2B} = 20 - 10 = 10\text{m}^3/\text{day};$$

$$t_{2A} = 50\text{day}; t_{2B} = 30\text{day}$$

$$= \frac{Q_{2,A}}{4\pi T} \ln \left(2.25 \frac{Tt_{2A}}{r^2S} \right) + \frac{Q_{2,B}}{4\pi T} \ln \left(2.25 \frac{Tt_{2B}}{r^2S} \right) =$$

$$= 0.080 + 0.076 = 0.156\text{m}$$

$$\Delta h_{tot} = 0.111 + 0.156 = 0.267\text{m}$$

b)

$$u = \frac{4Tt}{r^2S} = 8.89 \times 10^3 > 100 \quad \text{apply Cooper-Jacob}$$

$$\Delta h = \frac{Q_3}{4\pi T} \ln \left(2.25 \frac{Tt}{r^2S} \right) = 0.136 \text{ m}$$

9. From well P of Figure 8 a flow rate of 10 m³/day is extracted from a confined homogeneous and isotropic aquifer characterized by a thickness $B = 20$ m, a hydraulic conductivity $K=500$ cm/day and a specific elastic storage $S_s = 5 \times 10^{-6} \text{ m}^{-1}$.

- (a) how much is the drawdown in piezometer W after 10 days assuming the aquifer has an infinite horizontal dimension?
- (b) how much is the drawdown in piezometer W after 10 days if the boundary AB shown in the figure is assumed to be impermeable?
- (c) how much is the drawdown in piezometer W after 10 days if the boundary AB shown in the figure is assumed to be subject to a recharge flux so that a zero drawdown can be assumed?

Solution

- (a) The dimensionless Theis parameter u is:

$$u = \frac{4Tt}{r^2S}$$

where $r=100$ m, $T = kb=100 \text{ m}^2/\text{day}$ and $S = S_sb=10^{-4}$. We thus obtain $u = 4000 > 100$. Applying Cooper-Jacob approximation, the drawdown s is:

$$s_P = \frac{Q}{4\pi T} \ln \left(2.25 \frac{Tt}{r^2S} \right) = 6.1 \text{ cm}$$

- (b) Applying the superposition principle, and the image well technique we have that an impermeable boundary is (for symmetry reasons) equal to an image well P_1 located in a symmetric position with respect to P the other side of the boundary and with $Q_{P_1} = Q_P$. Thus the drawdown can be calculated as the sum of the two contributions $s_{tot} = s_P + s_{P_1}$ (where s_P is calculated above) using the distance between well W and well P_1 equal to $\sqrt{(400^2 + 100^2)} = 412.3$ m, to obtain $s_{P_1} = 3.9$ cm and $s_{tot} = 6.1 + 3.9 = 10$ cm.
- (c) Using the same technique as before, we need to put an image well in a symmetric location but now $Q_{P_1} = -Q_P$. Thus the total drawdown is in this case $s_{tot} = 6.1 - 3.9 = 2.2$ cm

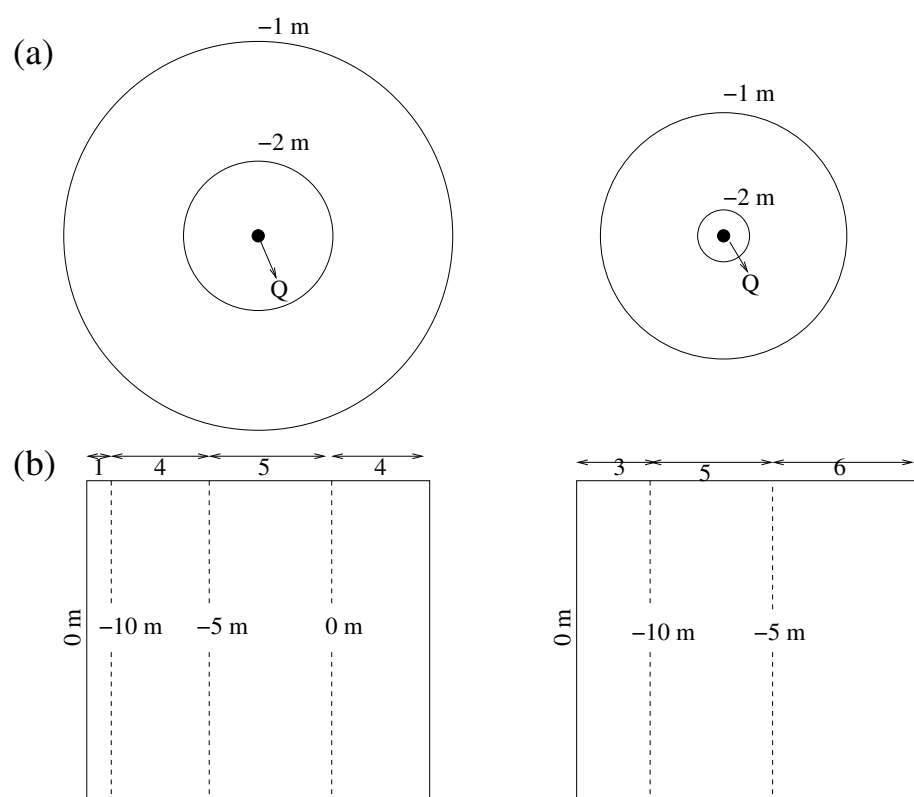


Figure 1: Exercise 2.

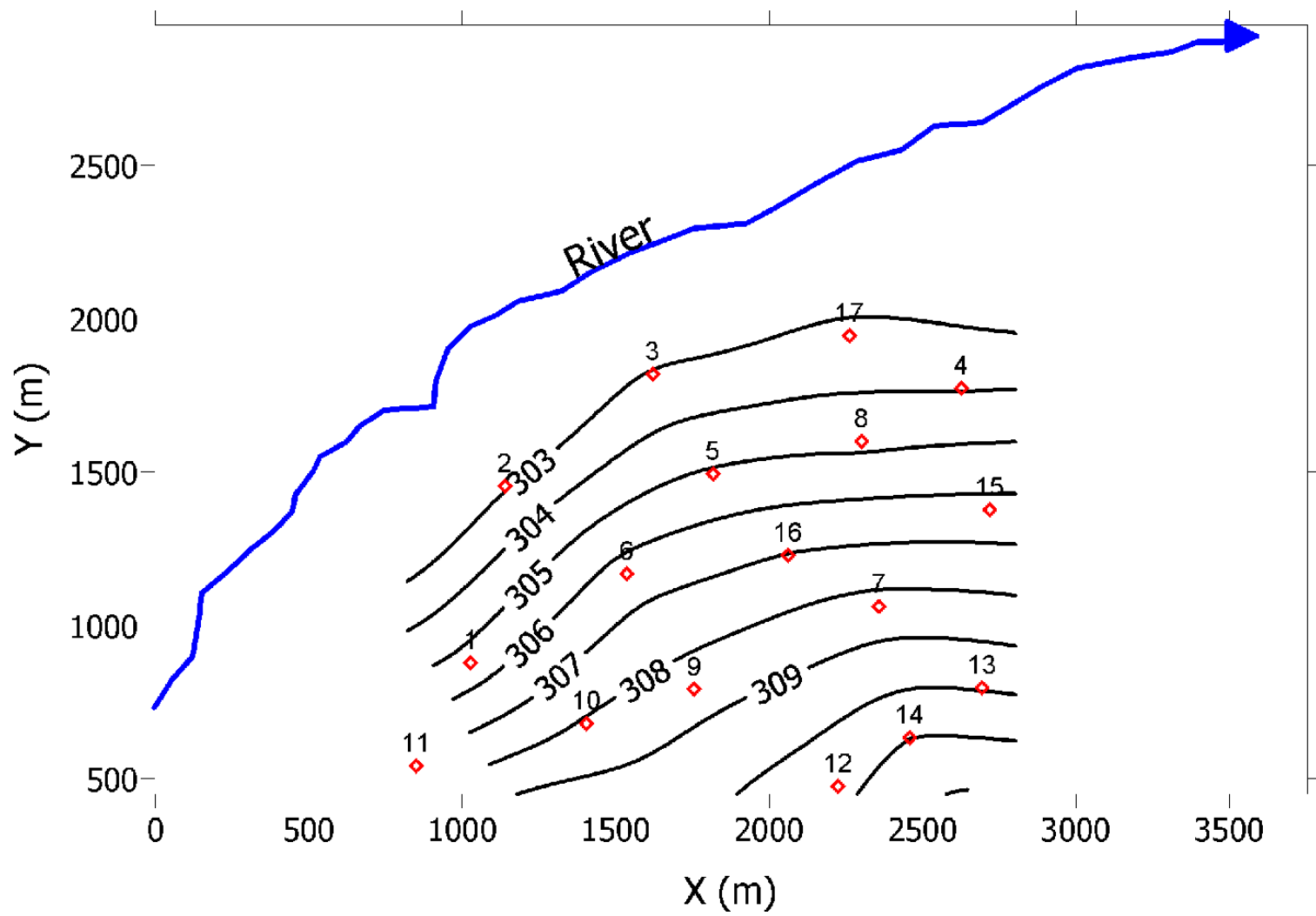


Figure 2: Exercise 3

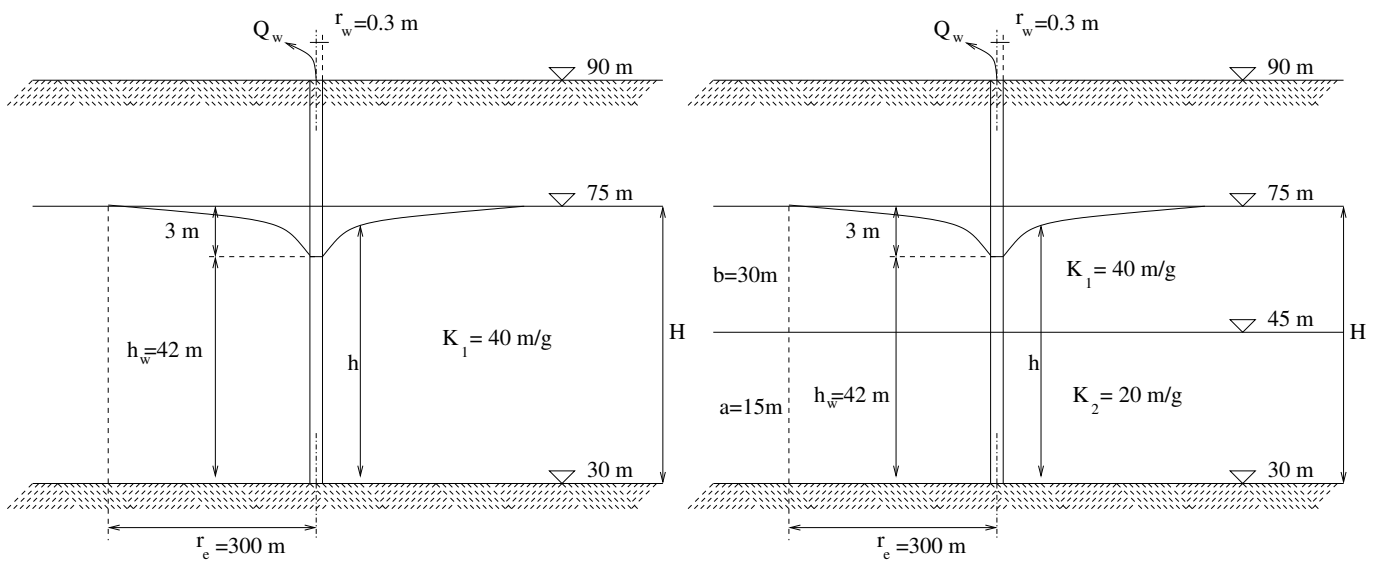


Figure 3: Exercise 4. the unit m/g is in Italian; it means m/day.

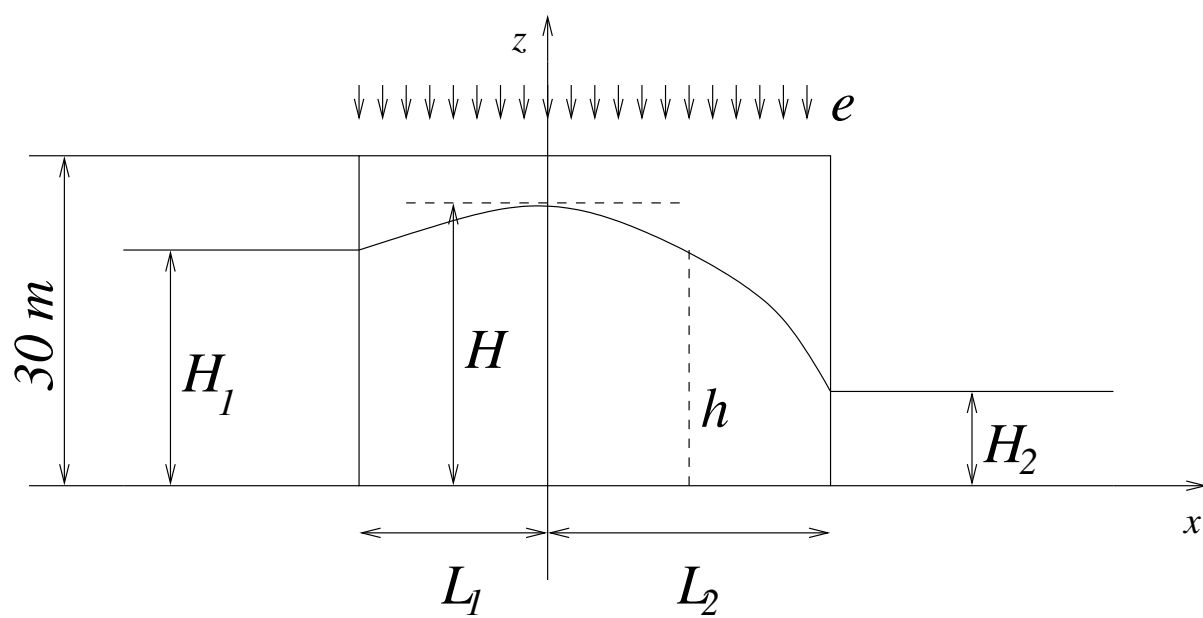


Figure 4: Exercise 5

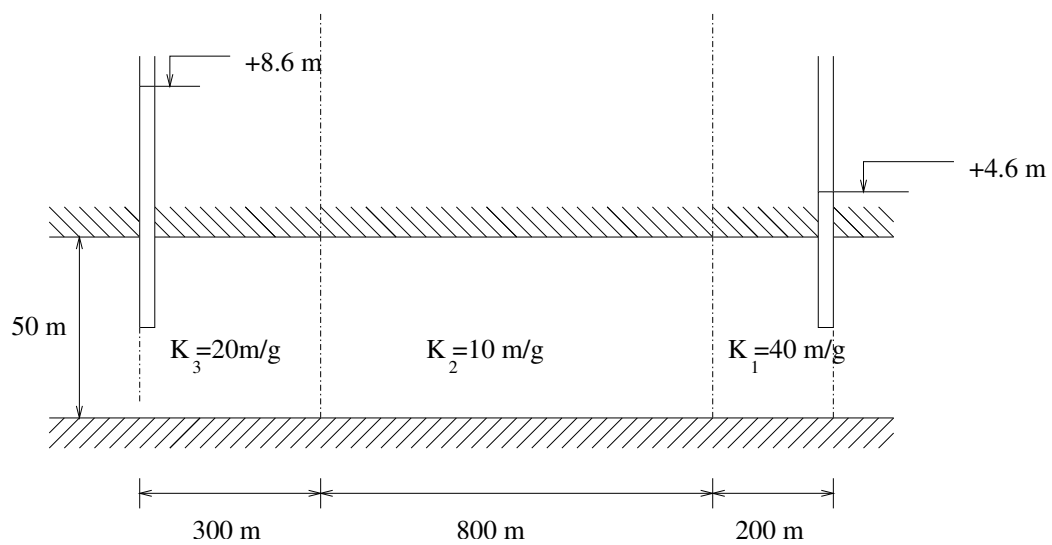


Figure 5: Exercise 6; the unit m/g is in Italian; it means m/day.

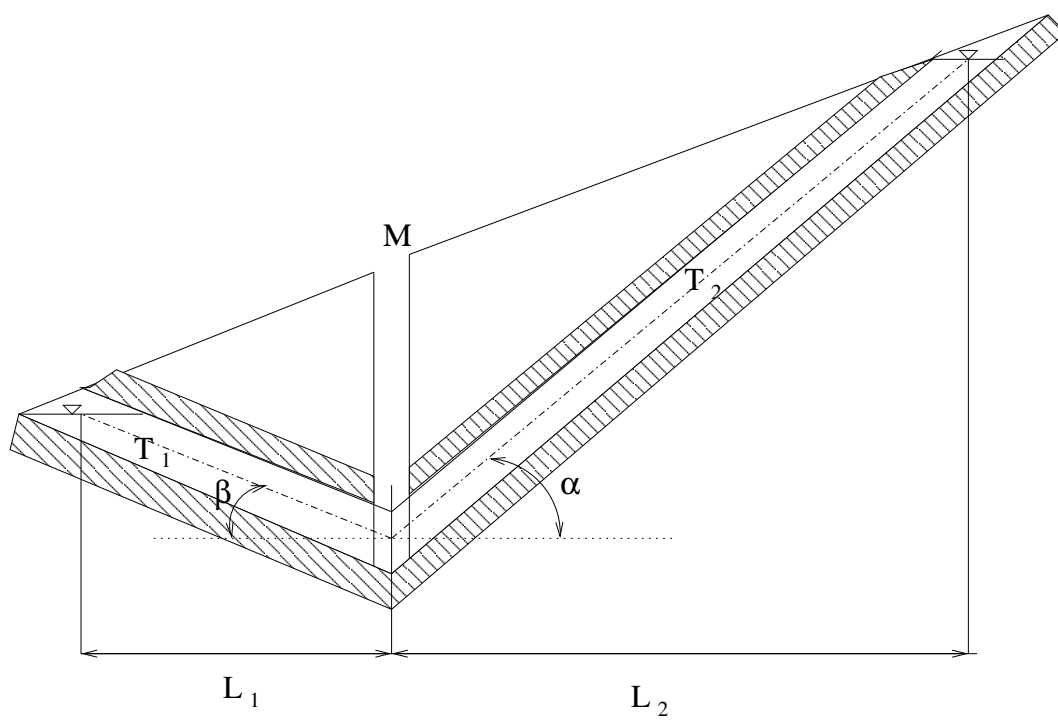


Figure 6: Exercise 7

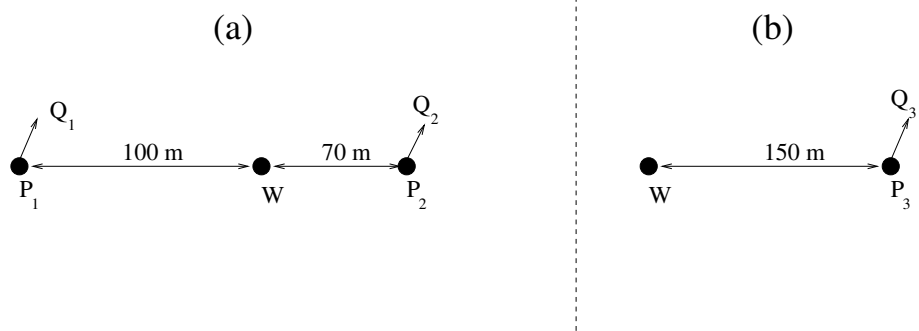


Figure 7: Exercise 8

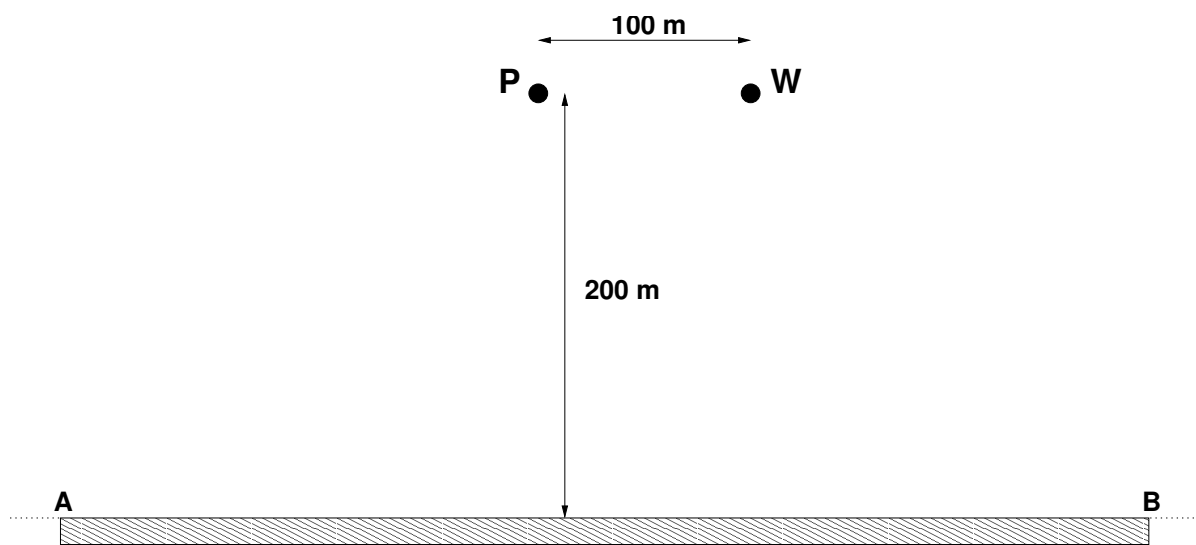


Figure 8: Exercise 9.