Flow in unsaturated porous medium

- **Motion equation**

1. Ignore the movement of water in the vapor phase
2. Ignore the movement due to pressure differences resulting from variations in salt concentration (*osmotic effect*)
3. Ignore the movement due to temperature variations (*thermo-osmotic effect*)
4. assume that the solid matrix is rigid and stable
5. the void space is occupied by a wetting (water) and a non-wetting phase (air)
6. the soil is isotropic

We shall consider flow resulting only from variations in piezometric head or in pressure in the water; the density ρ_w may vary.
The flow of the wetting and non-wetting phases can be described using the Darcy's law modified considering the permeability change owned to the presence of the other phase which occupies part of the void space.

$$q = -K \cdot \nabla\Phi = -\frac{k}{\mu}(\nabla p + \rho g \cdot 1 \cdot z) = -\frac{k}{\mu}\left(\frac{\partial p}{\partial x} \cdot 1 \cdot x + \frac{\partial p}{\partial y} \cdot 1 \cdot y + \left(\frac{\partial p}{\partial x} + \rho g\right) \cdot 1 \cdot z\right)$$

so

wetting phase: $$q_{iw} = -\frac{k_{ijw}(S_w)}{\mu_w}\left(\frac{\partial p_w}{\partial x_j} + \rho_w g \cdot \frac{\partial z}{\partial x_j}\right) = -k_{ij}\frac{k_{rw}(S_w)}{\mu_w}\left(\frac{\partial p_w}{\partial x_j} + \rho_w g \cdot \frac{\partial z}{\partial x_j}\right)$$

non-wetting phase: $$q_{inw} = -\frac{k_{ijnw}(S_{nw})}{\mu_{nw}}\left(\frac{\partial p_{nw}}{\partial x_j} + \rho_{nw} g \cdot \frac{\partial z}{\partial x_j}\right) = -k_{ij}\frac{k_{rnw}(S_{wn})}{\mu_{nw}}\left(\frac{\partial p_{nw}}{\partial x_j} + \rho_{nw} g \cdot \frac{\partial z}{\partial x_j}\right)$$

i =1,2,3

where:
k_{ijw}: unsaturated permeability of the wetting phase
k_{ijnw}: unsaturated permeability of the non-wetting phase
k_{ij}: permeability at saturation
k_{rw}: relative permeability of the wetting phase
k_{rnw}: relative permeability of the non-wetting phase

- **Mass conservation equation**

wetting phase: $$\frac{\partial(\rho_w n S_w)}{\partial t} + \nabla \cdot (\rho_w q_w) = 0$$

non-wetting phase: $$\frac{\partial(\rho_{nw} n S_{nw})}{\partial t} + \nabla \cdot (\rho_{nw} q_{nw}) = 0$$

- **Continuity equation**

The subsequent equations, valid for the

wetting phase: $$\frac{\partial(\rho_w n S_w)}{\partial t} - \frac{\partial}{\partial x_i} \cdot \left[\rho_w k_{ij} \frac{k_{rw}(S_w)}{\mu_w} \left(\frac{\partial p_w}{\partial x_j} + \rho_w g \cdot \frac{\partial z}{\partial x_j} \right) \right] = 0$$

non-wetting phase: $$\frac{\partial(\rho_{nw} n S_{nw})}{\partial t} - \frac{\partial}{\partial x_i} \cdot \left[\rho_{nw} k_{ij} \frac{k_{rnw}(S_{nw})}{\mu_{nw}} \left(\frac{\partial p_{nw}}{\partial x_j} + \rho_{nw} g \cdot \frac{\partial z}{\partial x_j} \right) \right] = 0$$

respectively, define the problem of the flow into an unsaturated porous medium.
The model is completed by a condition on the saturations of the wetting and non-wetting phases:

$$S_w + S_{nw} = 1$$

the relationships between the relative permeabilities of the wetting phase and the non-wetting one with the respective saturations

$$k_{rw} = k_{rw}(S_w)$$
$$k_{rnw} = k_{rnw}(S_{nw})$$

and of the moisture retention curve $S_w = S_w(p_c)$ where $p_c = p_{nw} - p_w$

Now let us simplify the problem considering only the motion in a vertical plane x-z and suppose that:

1. The non-wetting phase is at $p_{nw} = 0$
2. $\rho_w = \cos t$

The non-wetting phase doesn't move anymore and the motion equation for the wetting phase becomes:

$$q_w = -K_{Sw} k_{rw}(S_w) \nabla(\Psi_w + z)$$

where $\Psi_w = \dfrac{p_w}{\rho_w g}$

that is:

direction x: $$q_{xw} = -K_{Sw} k_{rw}(S_w) \left(\frac{1}{\rho_w g} \frac{\partial p_w}{\partial x} \right)$$

direction z: $$q_{zw} = -K_{Sw} k_{rw}(S_w) \left(\frac{\partial \Psi}{\partial z} + 1 \right)$$

where:

$K_{Sw} = \dfrac{k_w \rho_w g}{\mu_w}$: conductivity of wetting phase at saturation whereas the continuity equation becomes:

$$\frac{\partial (nS_w)}{\partial t} - \nabla \cdot [K_{Sw} k_{rw}(S_w) \nabla (\Psi_w + z)] = 0$$

that is

$$\frac{\partial (nS_w)}{\partial t} = \frac{\partial}{\partial x}\left[K_{Sw} k_{rw}(S_w)\frac{\partial \Psi_w}{\partial x}\right] + \frac{\partial}{\partial z}\left[K_{Sw} k_{rw}(S_w)\left(\frac{\partial \Psi_w}{\partial z} + 1\right)\right]$$

If you remind that the relationship between moisture content and saturation in water is: $\theta_w = n \cdot S_w$, the continuity equation for the vertical unsaturated porous media flow becomes:

$$\frac{\partial \theta_w}{\partial t} = \frac{\partial}{\partial x}\left[K_{Sw} k_{rw}(\theta_w)\frac{\partial \Psi_w}{\partial x}\right] + \frac{\partial}{\partial z}\left[K_{Sw} k_{rw}(\theta_w)\left(\frac{\partial \Psi_w}{\partial z} + 1\right)\right]$$

Recalling that saturation depends on capillary pressure, you can write $\frac{\partial \theta_w}{\partial t}$, generally called *general storage term*, as:

$$\frac{\partial \theta_w}{\partial t} = \frac{\partial \theta_w}{\partial \Psi_w} \cdot \frac{\partial \Psi_w}{\partial t} = \frac{\partial (nS_w)}{\partial \Psi_w} \cdot \frac{\partial \Psi_w}{\partial t} = \left[\frac{\partial n}{\partial \Psi_w} \cdot S_w + \frac{\partial S_w}{\partial \Psi_w} \cdot n\right] \cdot \frac{\partial \Psi_w}{\partial t} = \left[S_S \cdot S_w + \frac{\partial S_w}{\partial \Psi_w} \cdot n\right] \cdot \frac{\partial \Psi_w}{\partial t}$$

Now we can write the equation of vertical flow for an unsaturated medium known also as *Richards' equation:*

$$\boxed{\left[S_S \cdot S_w + \frac{\partial S_w}{\partial \Psi_w} \cdot n\right] \cdot \frac{\partial \Psi_w}{\partial t} = \frac{\partial}{\partial x}\left[K_{Sw} k_{rw}(\theta_w)\frac{\partial \Psi_w}{\partial x}\right] + \frac{\partial}{\partial z}\left[K_{Sw} k_{rw}(\theta_w)\left(\frac{\partial \Psi_w}{\partial z} + 1\right)\right]}$$