1.9 Normal and Hermitian Matrices

This section examines specific properties of normal and Hermitian matrices, including some optimality properties related to their spectra. The most common normal matrices that arise in practice are Hermitian and skew Hermitian.

1.9.1 Normal Matrices

By definition, a matrix is said to be normal if it commutes with its transpose conjugate, i.e., if it satisfies the relation

$$A^H A = AA^H. \tag{1.29}$$

An immediate property of normal matrices is stated in the following lemma.

Lemma 1.13. *If a normal matrix is triangular, then it is a diagonal matrix.*

Proof. Assume, for example, that A is upper triangular and normal. Compare the first diagonal element of the left-hand side matrix of (1.29) with the corresponding element of the matrix on the right-hand side. We obtain that

$$|a_{11}|^2 = \sum_{j=1}^{n} |a_{1j}|^2,$$

which shows that the elements of the first row are zeros except for the diagonal one. The same argument can now be used for the second row, the third row, and so on to the last row, to show that $a_{ij} = 0$ for $i \neq j$. □

A consequence of this lemma is the following important result.

Theorem 1.14. *A matrix is normal iff it is unitarily similar to a diagonal matrix.*

Proof. It is straightforward to verify that a matrix that is unitarily similar to a diagonal matrix is normal. We now prove that any normal matrix A is unitarily similar to a diagonal matrix. Let $A = QRQ^H$ be the Schur canonical form of A, where Q is unitary and R is upper triangular. By the normality of A,

$$QR^H Q^H QRQ^H = QRQ^H QR^H Q^H$$

or

$$QR^H RQ^H = QRR^H Q^H.$$

Upon multiplication by Q^H on the left and Q on the right, this leads to the equality $R^H R = RR^H$, which means that R is normal and, according to the previous lemma, this is only possible if R is diagonal. □

Thus, any normal matrix is diagonalizable and admits an orthonormal basis of eigenvectors, namely, the column vectors of Q.

The following result will be used in a later chapter. The question that is asked is, Assuming that any eigenvector of a matrix A is also an eigenvector of A^H, is A normal? If A has a full set of eigenvectors, then the result is true and easy to prove. Indeed, if V is the $n \times n$ matrix of common eigenvectors, then $AV = VD_1$ and $A^H V = VD_2$, with D_1 and D_2 diagonal. Then $AA^H V = VD_1D_2$ and $A^H AV = VD_2D_1$ and, therefore, $AA^H = A^H A$. It turns out that the result is true in general, i.e., independent of the number of eigenvectors that A admits.

Lemma 1.15. *A matrix A is normal iff each of its eigenvectors is also an eigenvector of A^H.*

Proof. If A is normal, then its left and right eigenvectors are identical, so the sufficient condition is trivial. Assume now that a matrix A is such that each of its eigenvectors v_i, $i = 1, \ldots, k$ with $k \le n$, is an eigenvector of A^H. For each eigenvector v_i of A, $Av_i = \lambda_i v_i$, and since v_i is also an eigenvector of A^H, then $A^H v_i = \mu v_i$. Observe that $(A^H v_i, v_i) = \mu(v_i, v_i)$ and, because $(A^H v_i, v_i) = (v_i, Av_i) = \bar{\lambda}_i(v_i, v_i)$, it follows that $\mu = \bar{\lambda}_i$. Next, it is proved by contradiction that there are no elementary divisors. Assume that the contrary is true for λ_i. Then the first principal vector u_i associated with λ_i is defined by

$$(A - \lambda_i I)u_i = v_i.$$

Taking the inner product of the above relation with v_i, we obtain

$$(Au_i, v_i) = \lambda_i(u_i, v_i) + (v_i, v_i). \tag{1.30}$$

On the other hand, it is also true that

$$(Au_i, v_i) = (u_i, A^H v_i) = (u_i, \bar{\lambda}_i v_i) = \lambda_i(u_i, v_i). \tag{1.31}$$

A result of (1.30) and (1.31) is that $(v_i, v_i) = 0$, which is a contradiction. Therefore, A has a full set of eigenvectors. This leads to the situation discussed just before the lemma, from which it is concluded that A must be normal. □

Clearly, Hermitian matrices are a particular case of normal matrices. Since a normal matrix satisfies the relation $A = QDQ^H$, with D diagonal and Q unitary, the eigenvalues of A are the diagonal entries of D. Therefore, if these entries are real it is clear that $A^H = A$. This is restated in the following corollary.

Corollary 1.16. *A normal matrix whose eigenvalues are real is Hermitian.*

As will be seen shortly, the converse is also true; i.e., a Hermitian matrix has real eigenvalues.

An eigenvalue λ of any matrix satisfies the relation

$$\lambda = \frac{(Au, u)}{(u, u)},$$

where u is an associated eigenvector. Generally, one might consider the complex scalars

$$\mu(x) = \frac{(Ax, x)}{(x, x)} \tag{1.32}$$

defined for any nonzero vector in $\mathbb{C}^n$. These ratios are known as *Rayleigh quotients* and are important for both theoretical and practical purposes. The set of all possible Rayleigh quotients as x runs over $\mathbb{C}^n$ is called the *field of values* of A. This set is clearly bounded since each $|\mu(x)|$ is bounded by the 2-norm of A; i.e., $|\mu(x)| \le \|A\|_2$ for all x.

If a matrix is normal, then any vector x in $\mathbb{C}^n$ can be expressed as

$$\sum_{i=1}^{n} \xi_i q_i,$$

where the vectors q_i form an orthogonal basis of eigenvectors, and the expression for $\mu(x)$ becomes

$$\mu(x) = \frac{(Ax,x)}{(x,x)} = \frac{\sum_{k=1}^{n} \lambda_k |\xi_k|^2}{\sum_{k=1}^{n} |\xi_k|^2} \equiv \sum_{k=1}^{n} \beta_k \lambda_k, \tag{1.33}$$

where

$$0 \le \beta_i = \frac{|\xi_i|^2}{\sum_{k=1}^{n} |\xi_k|^2} \le 1 \quad \text{and} \quad \sum_{i=1}^{n} \beta_i = 1.$$

From a well-known characterization of convex hulls established by Hausdorff (known as *Hausdorff's convex hull theorem*), this means that the set of all possible Rayleigh quotients as x runs over all of $\mathbb{C}^n$ is equal to the convex hull of the λ_i's. This leads to the following theorem, which is stated without proof.

Theorem 1.17. *The field of values of a normal matrix is equal to the convex hull of its spectrum.*

The next question is whether or not this is also true for nonnormal matrices—the answer is no: the convex hull of the eigenvalues and the field of values of a nonnormal matrix are different in general. As a generic example, one can take any nonsymmetric real matrix that has real eigenvalues only. In this case, the convex hull of the spectrum is a real interval but its field of values will contain imaginary values. See Exercise 12 for another example. It has been shown (by a theorem shown by Hausdorff) that the field of values of a matrix is a convex set. Since the eigenvalues are members of the field of values, their convex hull is contained in the field of values. This is summarized in the following proposition.

Proposition 1.18. *The field of values of an arbitrary matrix is a convex set that contains the convex hull of its spectrum. It is equal to the convex hull of the spectrum when the matrix is normal.*

A useful definition based on the field of values is that of the *numerical radius*. The numerical radius $\nu(A)$ of an arbitrary matrix A is the radius of the smallest disk containing the field of values; i.e.,

$$\nu(A) = \max_{x \in \mathbb{C}^n} |\mu(x)|.$$

It is easy to see that

$$\rho(A) \le \nu(A) \le \|A\|_2.$$

The spectral radius and numerical radius are identical for normal matrices. It can also be easily shown (see Exercise 21) that $\nu(A) \ge \|A\|_2/2$, which means that

$$\frac{\|A\|_2}{2} \le \nu(A) \le \|A\|_2. \tag{1.34}$$

The numerical radius is a vector norm; i.e., it satisfies (1.8)–(1.10), but it is not consistent (see Exercise 22). However, it satisfies the power inequality (see [171, p. 333]):

$$\nu(A^k) \le \nu(A)^k. \tag{1.35}$$

1.9.2 Hermitian Matrices

A first result on Hermitian matrices is the following.

Theorem 1.19. *The eigenvalues of a Hermitian matrix are real; i.e., $\sigma(A) \subset \mathbb{R}$.*

Proof. Let λ be an eigenvalue of A and u an associated eigenvector or 2-norm unity. Then

$$\lambda = (Au,u) = (u,Au) = \overline{(Au,u)} = \overline{\lambda},$$

which is the stated result. □

It is not difficult to see that if, in addition, the matrix is real, then the eigenvectors can be chosen to be real; see Exercise 24. Since a Hermitian matrix is normal, the following is a consequence of Theorem 1.14.

Theorem 1.20. *Any Hermitian matrix is unitarily similar to a real diagonal matrix.*

In particular, a Hermitian matrix admits a set of orthonormal eigenvectors that form a basis of $\mathbb{C}^n$.

In the proof of Theorem 1.17 we used the fact that the inner products (Au, u) are real. Generally, it is clear that any Hermitian matrix is such that (Ax, x) is real for any vector $x \in \mathbb{C}^n$. It turns out that the converse is also true; i.e., it can be shown that if (Az, z) is real for all vectors z in $\mathbb{C}^n$, then the matrix A is Hermitian (see Exercise 15).

Eigenvalues of Hermitian matrices can be characterized by optimality properties of the Rayleigh quotients (1.32). The best known of these is the min-max principle. We now label all the eigenvalues of A in descending order:

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n.$$

Here, the eigenvalues are not necessarily distinct and they are repeated, each according to its multiplicity. In the following theorem, known as the *min-max theorem*, S represents a generic subspace of $\mathbb{C}^n$.

Theorem 1.21. *The eigenvalues of a Hermitian matrix A are characterized by the relation*

$$\lambda_k = \min_{S,\ \dim(S)=n-k+1} \ \max_{x \in S, x \ne 0} \frac{(Ax,x)}{(x,x)}. \tag{1.36}$$

Proof. Let $\{q_i\}_{i=1,\ldots,n}$ be an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A associated with $\lambda_1, \ldots, \lambda_n$, respectively. Let S_k be the subspace spanned by the first k of these vectors and denote by $\mu(S)$ the maximum of $(Ax, x)/(x, x)$ over all nonzero vectors of a subspace S. Since the dimension of S_k is k, a well-known theorem of linear algebra shows that its intersection with any subspace S of dimension $n - k + 1$ is not reduced to $\{0\}$; i.e., there is a vector x in $S \bigcap S_k$. For this $x = \sum_{i=1}^{k} \xi_i q_i$, we have

$$\frac{(Ax, x)}{(x, x)} = \frac{\sum_{i=1}^{k} \lambda_i |\xi_i|^2}{\sum_{i=1}^{k} |\xi_i|^2} \geq \lambda_k,$$

so that $\mu(S) \geq \lambda_k$.

Consider, on the other hand, the particular subspace S_0 of dimension $n - k + 1$ that is spanned by $q_k, \ldots, q_n$. For each vector x in this subspace, we have

$$\frac{(Ax, x)}{(x, x)} = \frac{\sum_{i=k}^{n} \lambda_i |\xi_i|^2}{\sum_{i=k}^{n} |\xi_i|^2} \leq \lambda_k,$$

so that $\mu(S_0) \leq \lambda_k$. In other words, as S runs over all the $(n - k + 1)$-dimensional subspaces, $\mu(S)$ is never less than λ_k and there is at least one subspace S_0 for which $\mu(S_0) \leq \lambda_k$. This shows the desired result. ☐

The above result is often called the Courant–Fisher min-max principle or theorem. As a particular case, the largest eigenvalue of A satisfies

$$\lambda_1 = \max_{x \neq 0} \frac{(Ax, x)}{(x, x)}. \tag{1.37}$$

Actually, there are four different ways of rewriting the above characterization. The second formulation is

$$\lambda_k = \max_{S,\ \dim(S)=k}\ \min_{x \in S, x \neq 0} \frac{(Ax, x)}{(x, x)} \tag{1.38}$$

and the two other ones can be obtained from (1.36) and (1.38) by simply relabeling the eigenvalues increasingly instead of decreasingly. Thus, with our labeling of the eigenvalues in descending order, (1.38) tells us that the smallest eigenvalue satisfies

$$\lambda_n = \min_{x \neq 0} \frac{(Ax, x)}{(x, x)}, \tag{1.39}$$

with λ_n replaced by λ_1 if the eigenvalues are relabeled increasingly.

In order for all the eigenvalues of a Hermitian matrix to be positive, it is necessary and sufficient that

$$(Ax, x) > 0 \quad \forall\, x \in \mathbb{C}^n, \quad x \neq 0.$$

Such a matrix is called *positive definite*. A matrix that satisfies $(Ax, x) \geq 0$ for any x is said to be *positive semidefinite*. In particular, the matrix $A^H A$ is semipositive definite for any rectangular matrix, since

$$(A^H Ax, x) = (Ax, Ax) \geq 0 \quad \forall\, x.$$

Similarly, AA^H is also a Hermitian semipositive definite matrix. The square roots of the eigenvalues of $A^H A$ for a general rectangular matrix A are called the *singular values* of A and are denoted by σ_i. In Section 1.5, we stated without proof that the 2-norm of any matrix A is equal to the largest singular value σ_1 of A. This is now an obvious fact, because

$$\|A\|_2^2 = \max_{x \neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \max_{x \neq 0} \frac{(Ax, Ax)}{(x, x)} = \max_{x \neq 0} \frac{(A^H Ax, x)}{(x, x)} = \sigma_1^2,$$

which results from (1.37).

Another characterization of eigenvalues, known as the Courant characterization, is stated in the next theorem. In contrast with the min-max theorem, this property is recursive in nature.

Theorem 1.22. *The eigenvalue λ_i and the corresponding eigenvector q_i of a Hermitian matrix are such that*

$$\lambda_1 = \frac{(Aq_1, q_1)}{(q_1, q_1)} = \max_{x \in \mathbb{C}^n, x \neq 0} \frac{(Ax, x)}{(x, x)}$$

and, for $k > 1$,

$$\lambda_k = \frac{(Aq_k, q_k)}{(q_k, q_k)} = \max_{x \neq 0, q_1^H x = \cdots = q_{k-1}^H x = 0} \frac{(Ax, x)}{(x, x)}. \tag{1.40}$$

In other words, the maximum of the Rayleigh quotient over a subspace that is orthogonal to the first $k - 1$ eigenvectors is equal to λ_k and is achieved for the eigenvector q_k associated with λ_k. The proof follows easily from the expansion (1.33) of the Rayleigh quotient.

1.10 Nonnegative Matrices, *M*-Matrices

Nonnegative matrices play a crucial role in the theory of matrices. They are important in the study of convergence of iterative methods and arise in many applications, including economics, queuing theory, and chemical engineering.

A *nonnegative matrix* is simply a matrix whose entries are nonnegative. More generally, a partial order relation can be defined on the set of matrices.

Definition 1.23. *Let A and B be two $n \times m$ matrices. Then*

$$A \leq B$$

if, by definition, $a_{ij} \leq b_{ij}$ for $1 \leq i \leq n$, $1 \leq j \leq m$. If O denotes the $n \times m$ zero matrix, then A is nonnegative if $A \geq O$ and positive if $A > O$. Similar definitions hold in which "positive" is replaced by "negative."

The binary relation $\leq$ imposes only a *partial* order on $\mathbb{R}^{n \times m}$, since two arbitrary matrices in $\mathbb{R}^{n \times m}$ are not necessarily comparable by this relation. For the remainder of this section, we assume that only square matrices are involved. The next proposition lists a number of rather trivial properties regarding the partial order relation just defined.

Proposition 1.24. *The following properties hold:*

1. *The relation $\leq$ for matrices is reflexive ($A \leq A$), antisymmetric (if $A \leq B$ and $B \leq A$, then $A = B$), and transitive (if $A \leq B$ and $B \leq C$, then $A \leq C$).*
2. *If A and B are nonnegative, then so is their product AB and their sum $A + B$.*
3. *If A is nonnegative, then so is A^k.*
4. *If $A \leq B$, then $A^T \leq B^T$.*
5. *If $O \leq A \leq B$, then $\|A\|_1 \leq \|B\|_1$ and similarly $\|A\|_\infty \leq \|B\|_\infty$.*

The proof of these properties is left to Exercise 26.

A matrix is said to be *reducible* if there is a permutation matrix P such that PAP^T is block upper triangular. Otherwise, it is *irreducible*. An important result concerning nonnegative matrices is the following theorem known as the Perron–Frobenius theorem.

Theorem 1.25. *Let A be a real $n \times n$ nonnegative irreducible matrix. Then $\lambda \equiv \rho(A)$, the spectral radius of A, is a simple eigenvalue of A. Moreover, there exists an eigenvector u with positive elements associated with this eigenvalue.*

A relaxed version of this theorem allows the matrix to be reducible but the conclusion is somewhat weakened in the sense that the elements of the eigenvectors are only guaranteed to be *nonnegative*.

Next, a useful property is established.

Proposition 1.26. *Let A, B, C be nonnegative matrices, with $A \leq B$. Then*

$$AC \leq BC \quad \text{and} \quad CA \leq CB.$$

Proof. Consider the first inequality only, since the proof for the second is identical. The result that is claimed translates into

$$\sum_{k=1}^{n} a_{ik}c_{kj} \leq \sum_{k=1}^{n} b_{ik}c_{kj}, \quad 1 \leq i, j \leq n,$$

which is clearly true by the assumptions. □

A consequence of the proposition is the following corollary.

Corollary 1.27. *Let A and B be two nonnegative matrices, with $A \leq B$. Then*

$$A^k \leq B^k \quad \forall\, k \geq 0. \tag{1.41}$$

Proof. The proof is by induction. The inequality is clearly true for $k = 0$. Assume that (1.41) is true for k. According to the previous proposition, multiplying (1.41) from the left by A results in

$$A^{k+1} \leq AB^k. \tag{1.42}$$

Now, it is clear that if $B \geq 0$, then also $B^k \geq 0$, by Proposition 1.24. We now multiply both sides of the inequality $A \leq B$ by B^k to the right and obtain

$$AB^k \leq B^{k+1}. \tag{1.43}$$

The inequalities (1.42) and (1.43) show that $A^{k+1} \leq B^{k+1}$, which completes the induction proof. □

A theorem with important consequences on the analysis of iterative methods will now be stated.

Theorem 1.28. *Let A and B be two square matrices that satisfy the inequalities*

$$O \leq A \leq B. \tag{1.44}$$

Then

$$\rho(A) \leq \rho(B). \tag{1.45}$$

Proof. The proof is based on the following equality stated in Theorem 1.12:

$$\rho(X) = \lim_{k\to\infty} \|X^k\|^{1/k}$$

for any matrix norm. Choosing the 1-norm, for example, we have, from the last property in Proposition 1.24,

$$\rho(A) = \lim_{k\to\infty} \|A^k\|_1^{1/k} \leq \lim_{k\to\infty} \|B^k\|_1^{1/k} = \rho(B),$$

which completes the proof. □

Theorem 1.29. *Let B be a nonnegative matrix. Then $\rho(B) < 1$ iff $I - B$ is nonsingular and $(I - B)^{-1}$ is nonnegative.*

Proof. Define $C = I - B$. If it is assumed that $\rho(B) < 1$, then, by Theorem 1.11, $C = I - B$ is nonsingular and

$$C^{-1} = (I - B)^{-1} = \sum_{i=0}^{\infty} B^i. \tag{1.46}$$

In addition, since $B \geq 0$, all the powers of B as well as their sum in (1.46) are also nonnegative.

To prove the sufficient condition, assume that C is nonsingular and that its inverse is nonnegative. By the Perron–Frobenius theorem, there is a nonnegative eigenvector u associated with $\rho(B)$, which is an eigenvalue; i.e.,

$$Bu = \rho(B)u$$

or, equivalently,

$$C^{-1}u = \frac{1}{1 - \rho(B)}u.$$

Since u and C^{-1} are nonnegative and $I - B$ is nonsingular, this shows that $1 - \rho(B) > 0$, which is the desired result. □

Definition 1.30. *A matrix is said to be an M-matrix if it satisfies the following four properties:*

1. $a_{i,i} > 0$ *for* $i = 1, \ldots, n$.
2. $a_{i,j} \le 0$ *for* $i \neq j,\ i, j = 1, \ldots, n$.
3. *A is nonsingular.*
4. $A^{-1} \ge 0$.

In reality, the four conditions in the above definition are somewhat redundant and equivalent conditions that are more rigorous will be given later. Let A be any matrix that satisfies properties (1) and (2) in the above definition and let D be the diagonal of A. Since $D > 0$,

$$A = D - (D - A) = D\left(I - (I - D^{-1}A)\right).$$

Now define

$$B \equiv I - D^{-1}A.$$

Using the previous theorem, $I - B = D^{-1}A$ is nonsingular and $(I - B)^{-1} = A^{-1}D \ge 0$ iff $\rho(B) < 1$. It is now easy to see that conditions (3) and (4) of Definition 1.30 can be replaced with the condition $\rho(B) < 1$.

Theorem 1.31. *Let a matrix A be given such that*

1. $a_{i,i} > 0$ *for* $i = 1, \ldots, n$;
2. $a_{i,j} \le 0$ *for* $i \neq j,\ i, j = 1, \ldots, n$.

Then A is an M-matrix iff

3. $\rho(B) < 1$, *where* $B = I - D^{-1}A$.

Proof. From the above argument, an immediate application of Theorem 1.29 shows that properties (3) and (4) of Definition 1.30 are equivalent to $\rho(B) < 1$, where $B = I - C$ and $C = D^{-1}A$. In addition, C is nonsingular iff A is and C^{-1} is nonnegative iff A is. □

The next theorem shows that condition (1) of Definition 1.30 is implied by its other three conditions.

Theorem 1.32. *Let a matrix A be given such that*

1. $a_{i,j} \le 0$ *for* $i \neq j,\ i, j = 1, \ldots, n$;
2. *A is nonsingular;*
3. $A^{-1} \ge 0$.

Then

4. $a_{i,i} > 0$ *for* $i = 1, \ldots, n$; *i.e., A is an M-matrix;*
5. $\rho(B) < 1$, *where* $B = I - D^{-1}A$.

Proof. Define $C \equiv A^{-1}$. Writing that $(AC)_{ii} = 1$ yields

$$\sum_{k=1}^{n} a_{ik}c_{ki} = 1,$$

which gives

$$a_{ii}c_{ii} = 1 - \sum_{\substack{k=1 \\ k \neq i}}^{n} a_{ik}c_{ki}.$$

Since $a_{ik}c_{ki} \le 0$ for all k, the right-hand side is not less than 1 and, since $c_{ii} \ge 0$, then $a_{ii} > 0$. The second part of the result now follows immediately from an application of Theorem 1.31. □

Finally, this useful result follows.

Theorem 1.33. *Let A, B be two matrices that satisfy*

1. $A \le B$,
2. $b_{ij} \le 0$ *for all* $i \neq j$.

Then, if A is an M-matrix, so is the matrix B.

Proof. Assume that A is an M-matrix and let D_X denote the diagonal of a matrix X. The matrix D_B is positive because

$$D_B \ge D_A > 0.$$

Consider now the matrix $I - D_B^{-1}B$. Since $A \le B$, then

$$D_A - A \ge D_B - B \ge O,$$

which, upon multiplying through by D_A^{-1}, yields

$$I - D_A^{-1}A \ge D_A^{-1}(D_B - B) \ge D_B^{-1}(D_B - B) = I - D_B^{-1}B \ge O.$$

Since the matrices $I - D_B^{-1}B$ and $I - D_A^{-1}A$ are nonnegative, Theorems 1.28 and 1.31 imply that

$$\rho(I - D_B^{-1}B) \le \rho(I - D_A^{-1}A) < 1.$$

This establishes the result by using Theorem 1.31 once again. □

1.11 Positive Definite Matrices

A *real* matrix is said to be *positive definite* or *positive real* if

$$(Au, u) > 0 \quad \forall\, u \in \mathbb{R}^n,\ u \neq 0. \tag{1.47}$$

It must be emphasized that this definition is only useful when formulated entirely for real variables. Indeed, if u were not restricted to be real, then assuming that (Au, u) is real for all u complex would imply that A is Hermitian; see Exercise 15. If, in addition to the definition stated by (1.48), A is symmetric (real), then A is said to be *symmetric positive definite* (SPD). Similarly, if A is Hermitian, then A is said to be *Hermitian positive definite* (HPD). Some properties of HPD matrices were seen in Section 1.9, in particular with regard to their eigenvalues. Now the more general case where A is non-Hermitian and positive definite is considered.

We begin with the observation that any square matrix (real or complex) can be decomposed as

$$A = H + iS, \tag{1.48}$$

in which

$$H = \frac{1}{2}(A + A^H), \tag{1.49}$$

$$S = \frac{1}{2i}(A - A^H). \tag{1.50}$$

Note that both H and S are Hermitian while the matrix iS in the decomposition (1.48) is skew Hermitian. The matrix H in the decomposition is called the *Hermitian part* of A, while the matrix iS is the *skew-Hermitian part* of A. The above decomposition is the analogue of the decomposition of a complex number z into $z = x + iy$:

$$x = \Re e(z) = \frac{1}{2}(z + \bar{z}), \quad y = \Im m(z) = \frac{1}{2i}(z - \bar{z}).$$

When A is real and u is a real vector, then (Au, u) is real and, as a result, the decomposition (1.48) immediately gives the equality

$$(Au, u) = (Hu, u). \tag{1.51}$$

This results in the following theorem.

Theorem 1.34. *Let A be a real positive definite matrix. Then A is nonsingular. In addition, there exists a scalar $\alpha > 0$ such that*

$$(Au, u) \geq \alpha \|u\|_2^2 \tag{1.52}$$

for any real vector u.

Proof. The first statement is an immediate consequence of the definition of positive definiteness. Indeed, if A were singular, then there would be a nonzero vector such that $Au = 0$ and, as a result, $(Au, u) = 0$ for this vector, which would contradict (1.47). We now prove the second part of the theorem. From (1.51) and the fact that A is positive definite, we conclude that H is HPD. Hence, from (1.39), based on the min-max theorem, we get

$$\min_{u \neq 0} \frac{(Au, u)}{(u, u)} = \min_{u \neq 0} \frac{(Hu, u)}{(u, u)} \geq \lambda_{min}(H) > 0.$$

Taking $\alpha \equiv \lambda_{min}(H)$ yields the desired inequality (1.52). □

A simple yet important result that locates the eigenvalues of A in terms of the spectra of H and S can now be proved.

Theorem 1.35. *Let A be any square (possibly complex) matrix and let $H = \frac{1}{2}(A + A^H)$ and $S = \frac{1}{2i}(A - A^H)$. Then any eigenvalue λ_j of A is such that*

$$\lambda_{min}(H) \leq \Re e(\lambda_j) \leq \lambda_{max}(H), \tag{1.53}$$

$$\lambda_{min}(S) \leq \Im m(\lambda_j) \leq \lambda_{max}(S). \tag{1.54}$$

Proof. When the decomposition (1.48) is applied to the Rayleigh quotient of the eigenvector u_j associated with λ_j, we obtain

$$\lambda_j = (Au_j, u_j) = (Hu_j, u_j) + i(Su_j, u_j), \tag{1.55}$$

assuming that $\|u_j\|_2 = 1$. This leads to

$$\Re e(\lambda_j) = (Hu_j, u_j),$$
$$\Im m(\lambda_j) = (Su_j, u_j).$$

The result follows using properties established in Section 1.9. □

Thus, the eigenvalues of a matrix are contained in a rectangle defined by the eigenvalues of its Hermitian and non-Hermitian parts. In the particular case where A is real, then iS is skew Hermitian and its eigenvalues form a set that is symmetric with respect to the real axis in the complex plane. Indeed, in this case, iS is real and its eigenvalues come in conjugate pairs.

Note that all the arguments herein are based on the field of values and, therefore, they provide ways to localize the eigenvalues of A from knowledge of the field of values. However, this approximation can be inaccurate in some cases.

Example 1.3. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 10^4 & 1 \end{pmatrix}.$$

The eigenvalues of A are -99 and 101. Those of H are $1 \pm (10^4 + 1)/2$ and those of iS are $\pm i(10^4 - 1)/2$.

When a matrix B is SPD, the mapping

$$x, y \quad \rightarrow \quad (x, y)_B \equiv (Bx, y) \tag{1.56}$$

from $\mathbb{C}^n \times \mathbb{C}^n$ to $\mathbb{C}$ is a proper inner product on $\mathbb{C}^n$ in the sense defined in Section 1.4. The associated norm is often referred to as the *energy norm* or *A-norm*. Sometimes, it is possible to find an appropriate HPD matrix B that makes a given matrix A Hermitian, i.e., such that

$$(Ax, y)_B = (x, Ay)_B \quad \forall\, x, y,$$

although A is a non-Hermitian matrix with respect to the Euclidean inner product. The simplest examples are $A = B^{-1}C$ and $A = CB$, where C is Hermitian and B is HPD.

1.12 Projection Operators

Projection operators or *projectors* play an important role in numerical linear algebra, particularly in iterative methods for solving various matrix problems. This section introduces these operators from a purely algebraic point of view and gives a few of their important properties.

1.12.1 Range and Null Space of a Projector

A projector P is any linear mapping from $\mathbb{C}^n$ to itself that is idempotent, i.e., such that

$$P^2 = P.$$

A few simple properties follow from this definition. First, if P is a projector, then so is $(I - P)$, and the following relation holds:

$$\text{Ker}(P) = \text{Ran}(I - P). \tag{1.57}$$

In addition, the two subspaces Ker(P) and Ran(P) intersect only at the element zero. Indeed, if a vector x belongs to Ran(P), then $Px = x$ by the idempotence property. If it is also in Ker(P), then $Px = 0$. Hence, $x = Px = 0$, which proves the result. Moreover, every element of $\mathbb{C}^n$ can be written as $x = Px + (I - P)x$. Therefore, the space $\mathbb{C}^n$ can be decomposed as the direct sum

$$\mathbb{C}^n = \text{Ker}(P) \oplus \text{Ran}(P).$$

Conversely, every pair of subspaces M and S that forms a direct sum of $\mathbb{C}^n$ defines a unique projector such that Ran$(P) = M$ and Ker$(P) = S$. This associated projector P maps an element x of $\mathbb{C}^n$ into the component x_1, where x_1 is the M component in the unique decomposition $x = x_1 + x_2$ associated with the direct sum.

In fact, this association is unique; that is, an arbitrary projector P can be entirely determined by two subspaces: (1) the range M of P and (2) its null space S, which is also the range of $I - P$. For any x, the vector Px satisfies the conditions

$$Px \in M,$$
$$x - Px \in S.$$

The linear mapping P is said to project x *onto* M and *along* or *parallel to* the subspace S. If P is of rank m, then the range of $I - P$ is of dimension $n - m$. Therefore, it is natural to define S through its orthogonal complement $L = S^\perp$, which has dimension m. The above conditions, which define $u = Px$ for any x, become

$$u \in M, \tag{1.58}$$
$$x - u \perp L. \tag{1.59}$$

These equations define a projector P *onto* M and *orthogonal* to the subspace L. The first statement, (1.58), establishes the m degrees of freedom, while the second, (1.59), gives the m constraints that define Px from these degrees of freedom. The general definition of projectors is illustrated in Figure 1.1.

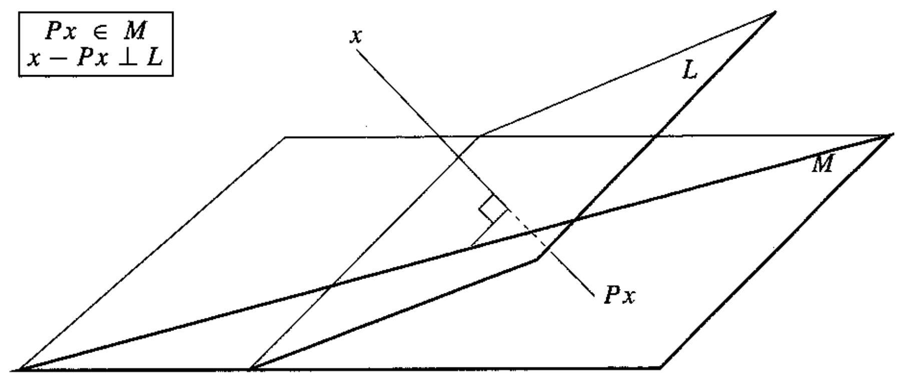

Figure 1.1. *Projection of x onto M and orthogonal to L.*

The question now is, Given two arbitrary subspaces M and L, both of dimension m, is it always possible to define a projector onto M orthogonal to L through the conditions (1.58) and (1.59)? The following lemma answers this question.

Lemma 1.36. *Given two subspaces M and L of the same dimension m, the following two conditions are mathematically equivalent:*

i. *No nonzero vector of M is orthogonal to L.*

ii. *For any x in $\mathbb{C}^n$ there is a unique vector u that satisfies the conditions* (1.58) *and* (1.59).

Proof. The first condition states that any vector that is in M and also orthogonal to L must be the zero vector. It is equivalent to the condition

$$M \cap L^\perp = \{0\}.$$

Since L is of dimension m, $L^\perp$ is of dimension $n - m$ and the above condition is equivalent to the condition that

$$\mathbb{C}^n = M \oplus L^\perp. \tag{1.60}$$

This in turn is equivalent to the statement that, for any x, there exists a unique pair of vectors u, w such that

$$x = u + w,$$

where u belongs to M and $w = x - u$ belongs to $L^\perp$, a statement that is identical to (ii). □

In summary, given two subspaces M and L satisfying the condition $M \cap L^\perp = \{0\}$, there is a projector P onto M orthogonal to L that defines the projected vector u of any vector x from (1.58) and (1.59). This projector is such that

$$\text{Ran}(P) = M, \qquad \text{Ker}(P) = L^\perp.$$

In particular, the condition $Px = 0$ translates into $x \in \text{Ker}(P)$, which means that $x \in L^\perp$. The converse is also true. Hence we have the following useful property:

$$Px = 0 \quad \text{iff} \quad x \perp L. \tag{1.61}$$

1.12.2 Matrix Representations

Two bases are required to obtain a matrix representation of a general projector: a basis $V = [v_1, \ldots, v_m]$ for the subspace $M = \text{Ran}(P)$ and a second one $W = [w_1, \ldots, w_m]$ for the subspace L. These two bases are *biorthogonal* when

$$(v_i, w_j) = \delta_{ij}. \tag{1.62}$$

In matrix form this means $W^H V = I$. Since Px belongs to M, let Vy be its representation in the V basis. The constraint $x - Px \perp L$ is equivalent to the condition

$$((x - Vy), w_j) = 0 \quad \text{for } j = 1, \ldots, m.$$

In matrix form, this can be rewritten as

$$W^H(x - Vy) = 0. \tag{1.63}$$

If the two bases are biorthogonal, then it follows that $y = W^H x$. Therefore, in this case, $Px = VW^H x$, which yields the matrix representation of P:

$$P = VW^H. \tag{1.64}$$

In case the bases V and W are not biorthogonal, then it is easily seen from the condition (1.63) that

$$P = V(W^H V)^{-1} W^H. \tag{1.65}$$

If we assume that no vector of M is orthogonal to L, then it can be shown that the $m \times m$ matrix $W^H V$ is nonsingular.

1.12.3 Orthogonal and Oblique Projectors

An important class of projectors is obtained in the case when the subspace L is equal to M, i.e., when

$$\text{Ker}(P) = \text{Ran}(P)^\perp.$$

Then the projector P is said to be the *orthogonal projector* onto M. A projector that is not orthogonal is *oblique*. Thus, an orthogonal projector is defined through the following requirements satisfied for any vector x:

$$Px \in M \quad \text{and} \quad (I - P)\,x \perp M \tag{1.66}$$

or, equivalently,

$$Px \in M \quad \text{and} \quad ((I - P)x, y) = 0 \quad \forall\, y \in M.$$

It is interesting to consider the mapping P^H defined as the adjoint of P:

$$(P^H x, y) = (x, Py) \quad \forall\, x,\ \forall\, y. \tag{1.67}$$

The above condition is illustrated in Figure 1.2.

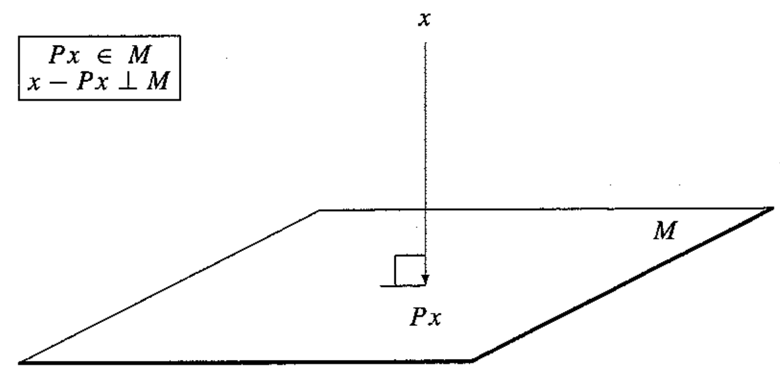

Figure 1.2. *Orthogonal projection of x onto a subspace M.*

First, note that P^H is also a projector because, for all x and y,

$$((P^H)^2 x, y) = (P^H x, Py) = (x, P^2 y) = (x, Py) = (P^H x, y).$$

A consequence of the relation (1.67) is

$$\text{Ker}(P^H) = \text{Ran}(P)^\perp, \tag{1.68}$$

$$\text{Ker}(P) = \text{Ran}(P^H)^\perp. \tag{1.69}$$

The above relations lead to the following proposition.

Proposition 1.37. *A projector is orthogonal iff it is Hermitian.*

Proof. By definition, an orthogonal projector is one for which $\text{Ker}(P) = \text{Ran}(P)^\perp$. Therefore, by (1.68), if P is Hermitian, then it is orthogonal. Conversely, if P is orthogonal, then (1.68) implies $\text{Ker}(P) = \text{Ker}(P^H)$ while (1.69) implies $\text{Ran}(P) = \text{Ran}(P^H)$. Since P^H is a projector and since projectors are uniquely determined by their range and null spaces, this implies that $P = P^H$. □

Given *any* unitary $n \times m$ matrix V whose columns form an orthonormal basis of $M = \text{Ran}(P)$, we can represent P by the matrix $P = VV^H$. This is a particular case of the matrix representation of projectors (1.64). In addition to being idempotent, the linear mapping associated with this matrix satisfies the characterization given above; i.e.,

$$VV^H x \in M \quad \text{and} \quad (I - VV^H)x \in M^\perp.$$

It is important to note that this representation of the orthogonal projector P is not unique. In fact, any orthonormal basis V will give a different representation of P in the above form. As a consequence, for any two orthogonal bases V_1, V_2 of M, we must have $V_1 V_1^H = V_2 V_2^H$, an equality that can also be verified independently; see Exercise 30.

1.12.4 Properties of Orthogonal Projectors

When P is an orthogonal projector, then the two vectors Px and $(I - P)x$ in the decomposition $x = Px + (I - P)x$ are orthogonal. The following relation results:

$$\|x\|_2^2 = \|Px\|_2^2 + \|(I - P)x\|_2^2.$$

A consequence of this is that, for any x,

$$\|Px\|_2 \le \|x\|_2.$$

Thus, the maximum of $\|Px\|_2/\|x\|_2$ for all x in $\mathbb{C}^n$ does not exceed one. In addition, the value one is reached for any element in $\mathrm{Ran}(P)$. Therefore,

$$\|P\|_2 = 1$$

for any orthogonal projector P.

An orthogonal projector has only two eigenvalues: zero or one. Any vector of the range of P is an eigenvector associated with the eigenvalue one. Any vector of the null space is obviously an eigenvector associated with the eigenvalue zero.

Next, an important optimality property of orthogonal projectors is established.

Theorem 1.38. *Let P be the orthogonal projector onto a subspace M. Then, for any given vector x in $\mathbb{C}^n$, the following is true:*

$$\min_{y \in M} \|x - y\|_2 = \|x - Px\|_2. \tag{1.70}$$

Proof. Let y be any vector of M and consider the square of its distance from x. Since $x - Px$ is orthogonal to M, to which $Px - y$ belongs, then

$$\|x - y\|_2^2 = \|x - Px + (Px - y)\|_2^2 = \|x - Px\|_2^2 + \|(Px - y)\|_2^2.$$

Therefore, $\|x - y\|_2 \ge \|x - Px\|_2$ for all y in M. Thus, we establish the result by noticing that the minimum is reached for $y = Px$. □

By expressing the conditions that define $y^* \equiv Px$ for an orthogonal projector P onto a subspace M, it is possible to reformulate the above result in the form of necessary and sufficient conditions that enable us to determine the best approximation to a given vector x in the least-squares sense.

Corollary 1.39. *Let a subspace M and a vector x in $\mathbb{C}^n$ be given. Then*

$$\min_{y \in M} \|x - y\|_2 = \|x - y^*\|_2 \tag{1.71}$$

iff the following two conditions are satisfied:

$$\begin{cases} y^* & \in & M, \\ x - y^* & \perp & M. \end{cases}$$

1.13 Basic Concepts in Linear Systems

Linear systems are among the most important and common problems encountered in scientific computing. From the theoretical point of view, it is well understood when a solution exists, when it does not, and when there are infinitely many solutions. In addition, explicit expressions of the solution using determinants exist. However, the numerical viewpoint is far more complex. Approximations may be available but it may be difficult to estimate how accurate they are. This clearly will depend on the data at hand, i.e., primarily the coefficient matrix. This section gives a very brief overview of the existence theory as well as the sensitivity of the solutions.

1.13.1 Existence of a Solution

Consider the *linear system*

$$Ax = b. \tag{1.72}$$

Here, x is termed the *unknown* and b the *right-hand side*. When solving the linear system (1.72), we distinguish three situations.

Case 1 The matrix A is nonsingular. There is a unique solution given by $x = A^{-1}b$.

Case 2 The matrix A is singular and $b \in \mathrm{Ran}(A)$. Since $b \in \mathrm{Ran}(A)$, there is an x_0 such that $Ax_0 = b$. Then $x_0 + v$ is also a solution for any v in $\mathrm{Ker}(A)$. Since $\mathrm{Ker}(A)$ is at least one-dimensional, there are infinitely many solutions.

Case 3 The matrix A is singular and $b \notin \mathrm{Ran}(A)$. There are no solutions.

Example 1.4. The simplest illustration of the above three cases is with small diagonal matrices. Let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 8 \end{pmatrix}.$$

Then A is nonsingular and there is a unique x given by

$$x = \begin{pmatrix} 0.5 \\ 2 \end{pmatrix}.$$

Now let

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then A is singular and, as is easily seen, $b \in \mathrm{Ran}(A)$. For example, a particular element x_0 such that $Ax_0 = b$ is $x_0 = \binom{0.5}{0}$. The null space of A consists of all vectors whose first component is zero, i.e., all vectors of the form $\binom{0}{\alpha}$. Therefore, there are infinitely many solutions given by

$$x(\alpha) = \begin{pmatrix} 0.5 \\ \alpha \end{pmatrix} \quad \forall \alpha.$$

Finally, let A be the same as in the previous case, but define the right-hand side as

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In this case there are no solutions because the second equation cannot be satisfied.

1.13.2 Perturbation Analysis

Consider the linear system (1.72), where A is an $n \times n$ nonsingular matrix. Given any matrix E, the matrix $A(\epsilon) = A + \epsilon E$ is nonsingular for ϵ small enough, i.e., for $\epsilon \leq \alpha$, where α is some small number; see Exercise 37. Assume that we perturb the data in the above system, i.e., that we perturb the matrix A by ϵE and the right-hand side b by ϵe. The solution $x(\epsilon)$ of the perturbed system satisfies the equation

$$(A + \epsilon E)x(\epsilon) = b + \epsilon e. \tag{1.73}$$

Let $\delta(\epsilon) = x(\epsilon) - x$. Then

$$\begin{aligned}(A + \epsilon E)\delta(\epsilon) &= (b + \epsilon e) - (A + \epsilon E)x \\ &= \epsilon(e - Ex), \\ \delta(\epsilon) &= \epsilon(A + \epsilon E)^{-1}(e - Ex).\end{aligned}$$

As an immediate result, the function $x(\epsilon)$ is differentiable at $\epsilon = 0$ and its derivative is given by

$$x'(0) = \lim_{\epsilon \to 0} \frac{\delta(\epsilon)}{\epsilon} = A^{-1}(e - Ex). \tag{1.74}$$

The size of the derivative of $x(\epsilon)$ is an indication of the size of the variation that the solution $x(\epsilon)$ undergoes when the data, i.e., the pair $[A, b]$, are perturbed in the direction $[E, e]$. In absolute terms, a small variation $[\epsilon E, \epsilon e]$ will cause the solution to vary by roughly $\epsilon x'(0) = \epsilon A^{-1}(e - Ex)$. The relative variation is such that

$$\frac{\|x(\epsilon) - x\|}{\|x\|} \leq \epsilon \|A^{-1}\| \left(\frac{\|e\|}{\|x\|} + \|E\| \right) + o(\epsilon).$$

Using the fact that $\|b\| \leq \|A\| \|x\|$ in the above equation yields

$$\frac{\|x(\epsilon) - x\|}{\|x\|} \leq \epsilon \|A\| \|A^{-1}\| \left(\frac{\|e\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right) + o(\epsilon), \tag{1.75}$$

which relates the relative variation in the solution to the relative sizes of the perturbations. The quantity

$$\kappa(A) = \|A\| \, \|A^{-1}\|$$

is called the *condition number* of the linear system (1.72) with respect to the norm $\|\cdot\|$. The condition number is relative to a norm. When using the standard norms $\|\cdot\|_p$, $p = 1, \ldots, \infty$, it is customary to label $\kappa(A)$ with the same label as the associated norm. Thus,

$$\kappa_p(A) = \|A\|_p \|A^{-1}\|_p.$$

For large matrices, the determinant of a matrix is almost never a good indication of "near" singularity or degree of sensitivity of the linear system. The reason is that $\det(A)$ is the product of the eigenvalues, which depends very much on a scaling of a matrix, whereas the condition number of a matrix is scaling invariant. For example, for $A = \alpha I$, the determinant is $\det(A) = \alpha^n$, which can be very small if $|\alpha| < 1$, whereas $\kappa(A) = 1$ for any of the standard norms.

In addition, small eigenvalues do not always give a good indication of poor conditioning. Indeed, a matrix can have all its eigenvalues equal to one yet be poorly conditioned.

Example 1.5. The simplest example is provided by matrices of the form

$$A_n = I + \alpha e_1 e_n^T$$

for large α. The inverse of A_n is

$$A_n^{-1} = I - \alpha e_1 e_n^T$$

and for the ∞-norm we have

$$\|A_n\|_\infty = \|A_n^{-1}\|_\infty = 1 + |\alpha|,$$

so that

$$\kappa_\infty(A_n) = (1 + |\alpha|)^2.$$

For a large α, this can give a very large condition number, whereas all the eigenvalues of A_n are equal to unity.

When an iterative procedure is used to solve a linear system, we typically face the problem of choosing a good stopping procedure for the algorithm. Often a residual norm,

$$\|r\| = \|b - A\tilde{x}\|,$$

is available for some current approximation $\tilde{x}$ and an estimate of the absolute error $\|x - \tilde{x}\|$ or the relative error $\|x - \tilde{x}\| / \|x\|$ is desired. The following simple relation is helpful in this regard:

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|b\|}.$$

It is necessary to have an estimate of the condition number $\kappa(A)$ in order to exploit the above relation.

Exercises

1. Verify that the Euclidean inner product defined by (1.4) does indeed satisfy the general definition of inner products on vector spaces.

2. Show that two eigenvectors associated with two distinct eigenvalues are linearly independent. In a more general sense, show that a family of eigenvectors associated with distinct eigenvalues forms a linearly independent family.

3. Show that, if λ is any nonzero eigenvalue of the matrix AB, then it is also an eigenvalue of the matrix BA. Start with the particular case where A and B are square and B is nonsingular, then consider the more general case where A, B may be singular or even rectangular (but such that AB and BA are square).

4. Let A be an $n \times n$ orthogonal matrix, i.e., such that $A^H A = D$, where D is a diagonal matrix. Assuming that D is nonsingular, what is the inverse of A? Assuming that $D > 0$, how can A be transformed into a unitary matrix (by operations on its rows or columns)?

5. Show that the Frobenius norm is consistent. Can this norm be associated with two vector norms via (1.7)? What is the Frobenius norm of a diagonal matrix? What is the p-norm of a diagonal matrix (for any p)?

6. Find the Jordan canonical form of the matrix

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Repeat the question for the matrix obtained by replacing the element a_{33} with 1.

7. Give an alternative proof of Theorem 1.9 on the Schur form by starting from the Jordan canonical form. [Hint: Write $A = XJX^{-1}$ and use the QR decomposition of X.]

8. Show from the definition of determinants used in Section 1.2 that the characteristic polynomial is a polynomial of degree n for an $n \times n$ matrix.

9. Show that the characteristic polynomials of two similar matrices are equal.

10. Show that

$$\lim_{k\to\infty} \|A^k\|^{1/k} = \rho(A)$$

for any matrix norm. [Hint: Use the Jordan canonical form.]

11. Let X be a nonsingular matrix and, for any matrix norm $\|\cdot\|$, define $\|A\|_X = \|AX\|$. Show that this is indeed a matrix norm. Is this matrix norm consistent? Show the same for $\|XA\|$ and $\|YAX\|$, where Y is also a nonsingular matrix. These norms are not, in general, associated with any vector norms; i.e., they can't be defined by a formula of the form (1.7). Why? What can you say when $Y = X^{-1}$? Is $\|X^{-1}AX\|$ associated with a vector norm in this particular case?

12. Find the field of values of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and verify that it is not equal to the convex hull of its eigenvalues.

13. Show that, for a skew-Hermitian matrix S,

$$\Re e(Sx, x) = 0 \quad \text{for any } x \in \mathbb{C}^n.$$

14. Given an arbitrary matrix S, show that, if $(Sx, x) = 0$ for all x in $\mathbb{C}^n$, then it is true that

$$(Sy, z) + (Sz, y) = 0 \quad \forall\, y,\ z \in \mathbb{C}^n. \tag{1.76}$$

[Hint: Expand $(S(y+z), y+z)$.]

15. Using the results of the previous two exercises, show that, if (Ax, x) is real for all x in $\mathbb{C}^n$, then A must be Hermitian. Would this result be true if the assumption were to be replaced with (Ax, x) *is real for all real* x? Explain.

16. Show that, if $(Sx, x) = 0$ for all complex vectors x, then S is zero. [Hint: Start by doing Exercise 14. Then, selecting $y = e_k$, $z = e^{\theta} e_j$ in (1.76) for an arbitrary θ, establish that $s_{kj} e^{2\theta} = -s_{jk}$ and conclude that $s_{jk} = s_{jk} = 0$.] Is the result true if $(Sx, x) = 0$ for all *real* vectors x?

17. The definition of a positive definite matrix is that (Ax, x) is real and positive for all real vectors x. Show that this is equivalent to requiring that the Hermitian part of A, namely, $\frac{1}{2}(A + A^H)$, be (Hermitian) positive definite.

18. Let $A_1 = B^{-1}C$ and $A_2 = CB$, where C is a Hermitian matrix and B is an HPD matrix. Are A_1 and A_2 Hermitian *in general*? Show that A_1 and A_2 are Hermitian (self-adjoint) with respect to the B inner product.

19. Let a matrix A be such that $A^H = p(A)$, where p is a polynomial. Show that A is normal. Given a diagonal complex matrix D, show that there exists a polynomial of degree less than n such that $\bar{D} = p(D)$. Use this to show that a normal matrix satisfies $A^H = p(A)$ for a certain polynomial of p of degree less than n. As an application, use this result to provide an alternative proof of Lemma 1.13.

20. Show that A is normal iff its Hermitian and skew-Hermitian parts, as defined in Section 1.11, commute.

21. The goal of this exercise is to establish the relation (1.34). Consider the numerical radius $\nu(A)$ of an arbitrary matrix A. Show that $\nu(A) \le \|A\|_2$. Show that, for a normal matrix, $\nu(A) = \|A\|_2$. Consider the decomposition of a matrix into its Hermitian and skew-Hermitian parts, as shown in (1.48), (1.49), and (1.50). Show that $\|A\|_2 \le \nu(H) + \nu(S)$. Now, using this inequality and the definition of the numerical radius, show that $\|A\|_2 \le 2\nu(A)$.

22. Show that the numerical radius is a vector norm in the sense that it satisfies the three properties (1.8)–(1.10) of norms. [Hint: For (1.8) solve Exercise 16 first.] Find a counter-example to show that the numerical radius is not a (consistent) matrix norm, i.e., that $\nu(AB)$ can be larger than $\nu(A)\nu(B)$.

23. Let A be a Hermitian matrix and B an HPD matrix defining a B inner product. Show that A is Hermitian (self-adjoint) with respect to the B inner product iff A and B commute. What condition must B satisfy for the same condition to hold in the more general case where A is not Hermitian?

24. Let A be a real symmetric matrix and λ an eigenvalue of A. Show that, if u is an eigenvector associated with λ, then so is $\bar{u}$. As a result, prove that, for any eigenvalue of a real symmetric matrix, there is an associated eigenvector that is real.

25. Show that a Hessenberg matrix H such that $h_{j+1,j} \ne 0$, $j = 1, 2, \ldots, n-1$, cannot be derogatory.

26. Prove all the properties listed in Proposition 1.24.

27. Let A be an M-matrix and u, v two nonnegative vectors such that $v^T A^{-1} u < 1$. Show that $A - uv^T$ is an M-matrix.

28. Show that if $O \le A \le B$, then $O \le A^T A \le B^T B$. Conclude that, under the same assumption, we have $\|A\|_2 \le \|B\|_2$.

29. Consider the subspace M of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}.$$

a. Write down the matrix representing the orthogonal projector onto M.

b. What is the null space of P?

c. What is its range?

d. Find the vector x in S that is the closest in the 2-norm sense to the vector $c = [1, 1, 1, 1]^T$.

30. Show that, for two orthonormal bases V_1, V_2 of the same subspace M of $\mathbb{C}^n$, we have $V_1 V_1^H x = V_2 V_2^H x \; \forall x$.

31. What are the eigenvalues of a projector? What about its eigenvectors?

32. Show that, if two projectors P_1 and P_2 commute, then their product $P = P_1 P_2$ is a projector. What are the range and kernel of P?

33. Theorem 1.32 shows that condition (2) in Definition 1.30 is not needed; i.e., it is implied by (4) (and the other conditions). One is tempted to say that only one of (2) or (4) is required. Is this true? In other words, does (2) also imply (4)? [Prove or show a counter-example.]

34. Consider the matrix A of size $n \times n$ and the vector $x \in \mathbb{R}^n$:

$$A = \begin{pmatrix} 1 & -1 & -1 & -1 & \dots & -1 \\ 0 & 1 & -1 & -1 & \dots & -1 \\ 0 & 0 & 1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 1 \\ 1/2 \\ 1/4 \\ 1/8 \\ \vdots \\ 1/2^{n-1} \end{pmatrix}.$$

a. Compute Ax, $\|Ax\|_2$, and $\|x\|_2$.

b. Show that $\|A\|_2 \ge \sqrt{n}$.

c. Give a lower bound for $\kappa_2(A)$.

35. What is the inverse of the matrix A of the previous exercise? Give an expression for $\kappa_1(A)$ and $\kappa_\infty(A)$ based on this.

36. Find a small rank-one perturbation that makes the matrix A in Exercise 34 singular. Derive a lower bound for the singular values of A.

37. Consider a nonsingular matrix A. Given any matrix E, show that there exists α such that the matrix $A(\epsilon) = A + \epsilon E$ is nonsingular for all $\epsilon < \alpha$. What is the largest possible value for α satisfying the condition? [Hint: Consider the eigenvalues of the generalized eigenvalue problem $Au = \lambda Eu$.]

Notes and References

For additional reading on the material presented in this chapter, see Golub and Van Loan [149], Meyer [209], Demmel [99], Datta [93], Stewart [272], and Varga [292]. Volume 2 ("Eigensystems") of the series [273] offers up-to-date coverage of algorithms for eigenvalue problems. The excellent treatise of nonnegative matrices in the book by Varga [292] remains a good reference on this topic and on iterative methods four decades after its first publication. State-of-the-art coverage of iterative methods up to the beginning of the 1970s can be found in the book by Young [321], which covers M-matrices and related topics in great detail. For a good overview of the linear algebra aspects of matrix theory and a complete proof of Jordan's canonical form, Halmos [164] is recommended.