

# Esercizio degli EF triangolari

17a

$$\Omega(u) = \iint_R \frac{1}{2} \left\{ T_x \left( \frac{\partial u}{\partial x} \right)^2 + T_y \left( \frac{\partial u}{\partial y} \right)^2 \right\} dx dy + \iint_R f u dx dy$$

Si minimizza  $\Omega(u_n)$  con  $u_n = \sum_{i=1}^n a_i \xi_i(x,y)$

$$\frac{\partial \Omega(u_n)}{\partial a_i} = \iint_R \left\{ T_x \left( \frac{\partial u_n}{\partial x} \right) \frac{\partial \xi_i}{\partial x} + T_y \left( \frac{\partial u_n}{\partial y} \right) \frac{\partial \xi_i}{\partial y} \right\} dx dy + \iint_R f \left( \frac{\partial u_n}{\partial a_i} \right) dx dy =$$

$$= \iint_R \left\{ T_x \left( a_1 \frac{\partial \xi_1}{\partial x} + a_2 \frac{\partial \xi_2}{\partial x} + \dots + a_n \frac{\partial \xi_n}{\partial x} \right) \frac{\partial \xi_i}{\partial x} + \dots \right. \\ \left. + T_y \left( a_1 \frac{\partial \xi_1}{\partial y} + a_2 \frac{\partial \xi_2}{\partial y} + \dots + a_n \frac{\partial \xi_n}{\partial y} \right) \frac{\partial \xi_i}{\partial y} \right\} dx dy + \iint_R f \xi_i(x,y) dx dy = 0$$

Questa è l'equazione  $i$ -esima di Ritz - Galerkin  
a cui si aggiunge il termine noto

$$\iint_R f \xi_i dx dy$$

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se il problema è di evoluzione il principio variazionale ristretto è:

$$\delta U(u) = \iint_R \frac{1}{2} \left\{ T_x \left( \frac{\partial u}{\partial x} \right)^2 + T_y \left( \frac{\partial u}{\partial y} \right)^2 \right\} dx dy + \iint_R \left[ f + \rho \frac{\partial u}{\partial t} \right] dx dy$$

$$\text{Si pone } u_n(x, y, t) = \sum_{i=1}^n a_i(t) \xi_i(x, y)$$

$$\frac{\partial u_n}{\partial t} = \sum_{i=1}^n \frac{\partial a_i}{\partial t} \xi_i(x, y)$$

Pertanto il contributo dell'  $\iint_R \rho u_n \left[ \frac{\partial u_n}{\partial t} \right] dx dy$  è:

$$\begin{aligned} \frac{\partial}{\partial a_i} \left[ \iint_R \rho u_n \left[ \frac{\partial u_n}{\partial t} \right] dx dy \right] &= \\ &= \iint_R \rho \xi_i(x, y) \left[ \frac{\partial a_1}{\partial t} \xi_1 + \frac{\partial a_2}{\partial t} \xi_2 + \dots + \frac{\partial a_n}{\partial t} \xi_n \right] dx dy \end{aligned}$$

Questo contributo si aggiunge all'ep.  $i$ -esima

Termine noto dell'ep.  $i$ -esima è sempre

$$\iint_R \rho \xi_i dx dy$$

$$[H]\{a\} + \{q\} = 0$$

Eq. di filtrazione:

$$h_{ij} = \int_R \left( T_x \frac{\partial \xi_i}{\partial x} \frac{\partial \xi_j}{\partial x} + T_y \frac{\partial \xi_i}{\partial y} \frac{\partial \xi_j}{\partial y} \right) dx dy$$

Problem: time-dependent:

$$u_n(x,t) = \sum_{i=1}^n a_i(t) \xi_i(x)$$

$$\frac{\partial u_n}{\partial t} = \sum_{i=1}^n \frac{\partial a_i}{\partial t} \xi_i(x)$$

$$[H]\{a\} + [P]\left\{\frac{\partial a}{\partial t}\right\} + \{q\} = 0 \quad \text{lineare}$$

$$P_{ij} = \int_S \xi_i \xi_j dx dy$$

$$q_i = \int_R f \xi_i dx dy$$

# Applicazioni

$$\frac{\partial}{\partial x} \left( k_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial h}{\partial y} \right) = S \frac{\partial h}{\partial t} + f(x, y, t) \quad \text{in } \Omega$$

Condizioni di Neumann su  $\partial R_2$ :

$$k_x \frac{\partial h}{\partial x} n_x + k_y \frac{\partial h}{\partial y} n_y = q$$

+ Dirichlet.

Funzionale associato:

$$\Omega(h) = \iint_R \left\{ \frac{1}{2} \left[ k_x \left( \frac{\partial h}{\partial x} \right)^2 + k_y \left( \frac{\partial h}{\partial y} \right)^2 \right] + \left( S \frac{\partial h}{\partial t} + f \right) h \right\} dx dy - \int_{\partial R_2} q h d\Gamma$$

Elementi finiti triangolari:

$$h^e = \xi_i^e(x, y) h_i(t) + \xi_j^e(x, y) h_j(t) + \xi_m^e(x, y) h_m(t)$$

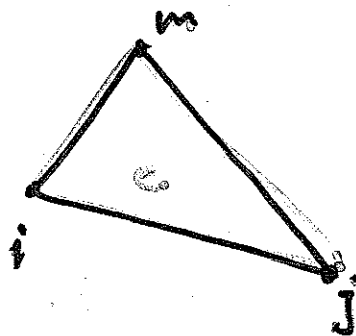
$$\xi_i^e(x, y) = (a_i + b_i x + c_i y) / 2\Delta$$

$$a_i = x_j y_m - x_m y_j$$

$$b_i = y_j - y_m$$

$$c_i = x_m - x_j$$

$\Delta =$  area del triangolo  $ijm$



$$\Omega(h) = \sum_e \Omega^e(h)$$

Contributo di e alle 3 derivate:

$$\frac{\partial \Omega^e}{\partial h_i}$$

$$\frac{\partial \Omega^e}{\partial h_j}$$

$$\frac{\partial \Omega^e}{\partial h_m}$$

$$\begin{aligned} \Omega^e = \iint_{\Delta_e} & \left\{ \frac{1}{2} \left[ k_x^e \left( \frac{\partial \xi_i^e}{\partial x} h_i + \frac{\partial \xi_j^e}{\partial x} h_j + \frac{\partial \xi_m^e}{\partial x} h_m \right) + \right. \right. \\ & \left. \left. + k_y^e \left( \frac{\partial \xi_i^e}{\partial y} h_i + \frac{\partial \xi_j^e}{\partial y} h_j + \frac{\partial \xi_m^e}{\partial y} h_m \right) \right]^2 + \right. \\ & \left. + \left[ S^e \left( \xi_i^e \frac{\partial h_i}{\partial t} + \xi_j^e \frac{\partial h_j}{\partial t} + \xi_m^e \frac{\partial h_m}{\partial t} \right) + f^e \right] \left( \xi_i^e h_i + \xi_j^e h_j + \xi_m^e h_m \right) \right\} dx dy \end{aligned}$$

$$\begin{aligned} \frac{\partial \Omega^e}{\partial h_i} = \iint_{\Delta_e} & \left\{ k_x^e \left( \frac{\partial \xi_i^e}{\partial x} h_i + \frac{\partial \xi_j^e}{\partial x} h_j + \frac{\partial \xi_m^e}{\partial x} h_m \right) \frac{\partial \xi_i^e}{\partial x} + \right. \\ & \left. + k_y^e \left( \frac{\partial \xi_i^e}{\partial y} h_i + \frac{\partial \xi_j^e}{\partial y} h_j + \frac{\partial \xi_m^e}{\partial y} h_m \right) \frac{\partial \xi_i^e}{\partial y} + \right. \\ & \left. + \left[ S^e \left( \xi_i^e \frac{\partial h_i}{\partial t} + \xi_j^e \frac{\partial h_j}{\partial t} + \xi_m^e \frac{\partial h_m}{\partial t} \right) + f^e \right] \xi_i^e \right\} dx dy \end{aligned}$$

Esprimendo le derivate delle  $\xi_i$ :

$$\begin{aligned} \frac{\partial \Omega^e}{\partial h_i} = \iint_{\Delta_e} & \frac{1}{2\Delta} \left\{ k_x^e (b_i h_i + b_j h_j + b_m h_m) \frac{b_i}{2\Delta} + \right. \\ & \left. + k_y^e (c_i h_i + c_j h_j + c_m h_m) \frac{c_i}{2\Delta} \right\} dx dy + \\ & + \iint_{\Delta_e} \left[ S^e \left( \xi_i^e \frac{\partial h_i}{\partial t} + \xi_j^e \frac{\partial h_j}{\partial t} + \xi_m^e \frac{\partial h_m}{\partial t} \right) + f^e \right] \xi_i^e dx dy \end{aligned}$$

Matrice di rigidità (locale):

$$H_{ik}^e = \frac{1}{4\Delta^2} (k_x^e b_i b_k + k_y^e c_i c_k) \quad k = i, j, m$$

Matrice di capacità (locale):

$$P_{ij}^e = \iint_{\Delta^e} \xi_i^e \xi_j^e dx dy = \frac{\Delta}{12} \int_{\Delta^e} \begin{cases} 2 & i=j \\ 1 & i \neq j \end{cases}$$

$$q_i = \iint_{\Delta^e} f^e \xi_i^e dx dy = \frac{\Delta}{3} f^e$$

Se  $i$  è nodo del contorno si ha:

$$\frac{\partial}{\partial h_i} \int_{\Gamma^e} q^e h^e d\Gamma = \int_{\Gamma^e} q^e \xi_i^e d\Gamma = q^e \frac{l^e}{2}$$

Infine si scrive:

$$[H] \{h\} + [P] \left\{ \frac{\partial h}{\partial t} \right\} + \{q\} = 0$$

$$\{h\} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{bmatrix}$$

Stazionario:

$$\frac{\partial}{\partial x} \left( k_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_y \frac{\partial h}{\partial y} \right) = f(x, y)$$

$\Downarrow$

$$[H] \{h\} + \{q\} = 0$$

$[H]$  e  $[P]$  sono matrici simmetriche d.p.

$$\Omega(h) = \frac{1}{2} \{h\}^T [H] \{h\} > 0$$

Equazione di diffusione - convezione:

$$\frac{\partial}{\partial x} \left( D_x \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial y} \left( D_y \frac{\partial c}{\partial y} \right) - \left( v_x \frac{\partial c}{\partial x} + v_y \frac{\partial c}{\partial y} \right) = \frac{\partial c}{\partial t}$$

( con  $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$   
cioè campo di moto stazionario )

Applichiamo l'approccio variazionale di Galerkin:

$$H_{ij} = \iint_R \left\{ D_x \frac{\partial \xi_i}{\partial x} \frac{\partial \xi_j}{\partial x} + D_y \frac{\partial \xi_i}{\partial y} \frac{\partial \xi_j}{\partial y} \right\} dx dy$$

$$P_{ij} = \iint_R \xi_i \xi_j dx dy$$

$$B_{ij} = \iint_R \left( v_x \frac{\partial \xi_j}{\partial x} + v_y \frac{\partial \xi_j}{\partial y} \right) \xi_i dx dy$$

$$B_{ij} \neq B_{ji}$$

$$c_n = c_1 \xi_1(x, y) + c_2 \xi_2(x, y) + \dots + c_n \xi_n(x, y)$$

Se le velocità sono costanti nel triangolo si ha:

$$B_{ij}^e = \iint_{\Delta_e} \frac{1}{2\Delta} (v_x^e b_j + v_y^e c_j) \left( \frac{a_i + b_i x + c_i y}{2\Delta} \right) dx dy$$
$$= \frac{1}{6} (v_x^e b_j + v_y^e c_j)$$

$$B_{ij}^e \neq B_{ji}^e$$

Dal procedimento scaturisce l'equazione:

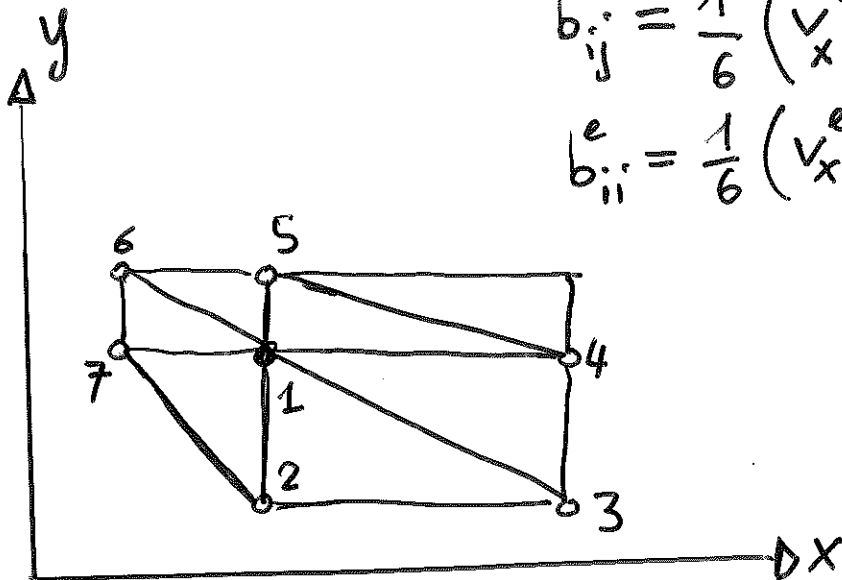
$$([H] + [B]) \{c\} + [P] \left\{ \frac{\partial c}{\partial t} \right\} + \{q\} = 0$$

La matrice  $[H] + [B]$  non è simmetrica

ancorché sparsa. Il GCM in genere non converge. Si possono usare altri metodi del gradiente opportunamente preconditionato:

- 1- GMRES (Generalized Minimal Residual)
- 2- BiCGSTAB (Bi Conjugate Gradient Stabilized)
- 3- TFQMR (Transpose Free Quasi Minimal Residual)





$$b_{ij}^e = \frac{1}{6} (v_x^e b_j^e + v_y^e c_j^e)$$

$$b_{ii}^e = \frac{1}{6} (v_x^e b_i^e + v_y^e c_i^e)$$

Assumiamo componenti della velocità  $v_x$  e  $v_y$  uniformi sul patch di elementi che fanno capo al nodo 1. Si ha:

$$b_{1,1} = \frac{1}{6} \left\{ v_x \left[ \frac{(y_4 - y_5)}{2} + \frac{(y_5 - y_6)}{2} + \frac{(y_6 - y_7)}{2} + \frac{(y_7 - y_2)}{2} + \frac{(y_2 - y_3)}{2} + \frac{(y_3 - y_4)}{2} \right] + v_y \left[ \frac{(x_5 - x_4)}{2} + \frac{(x_6 - x_5)}{2} + \frac{(x_7 - x_6)}{2} + \frac{(x_2 - x_7)}{2} + \frac{(x_3 - x_2)}{2} + \frac{(x_4 - x_3)}{2} \right] \right\} = 0$$

$b_{1,j} = -b_{j,1}$ . Ad esempio:

$$\left\{ \begin{aligned} b_{14} &= \frac{1}{6} \left[ v_x (y_1 - y_3) + v_y (x_3 - x_1) + v_x (y_5 - y_1) + v_y (x_1 - x_5) \right] \\ &= \frac{1}{6} \left[ v_x (y_5 - y_3) + v_y (x_3 - x_5) \right] \end{aligned} \right.$$

$$\left\{ \begin{aligned} b_{41} &= \frac{1}{6} \left[ v_x (y_4 - y_5) + v_y (x_5 - x_4) + v_x (y_3 - y_4) + v_y (x_4 - x_3) \right] \\ &= \frac{1}{6} \left[ v_x (y_3 - y_5) + v_y (x_5 - x_3) \right] = -b_{14} \end{aligned} \right.$$

$$A_s = \frac{A + A^T}{2}$$

$$A_{-s} = \frac{A - A^T}{2}$$

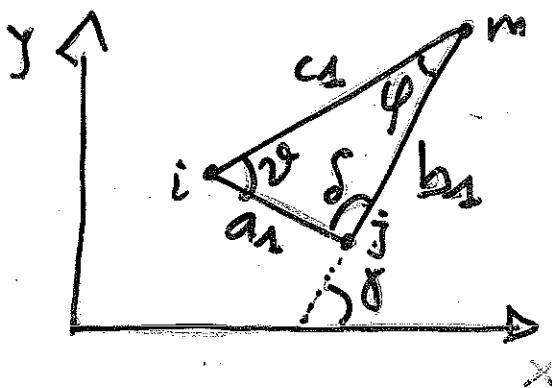
$$A = A_s + A_{-s}$$

17-8bis

# Problema isotropo ( $k_x = k_y$ )

La matrice locale di rigidità è:

$$H^e = \frac{k^e}{4\Delta^e} \begin{bmatrix} b_i^2 + c_i^2 & b_i b_j + c_i c_j & b_i b_m + c_i c_m \\ b_j b_i + c_j c_i & b_j^2 + c_j^2 & b_j b_m + c_j c_m \\ b_m b_i + c_m c_i & b_m b_j + c_m c_j & b_m^2 + c_m^2 \end{bmatrix}$$



$$\begin{aligned} b_i &= y_j - y_m = -b_1 \sin \gamma & c_i &= x_m - x_j = b_1 \cos \gamma \\ b_j &= y_m - y_i = -c_1 \sin(\gamma - \varphi) & c_j &= x_i - x_m = -c_1 \cos(\gamma - \varphi) \\ b_m &= y_i - y_j = a_1 \sin(\gamma + \delta) & c_m &= x_j - x_i = -a_1 \cos(\gamma + \delta) \end{aligned}$$

Risulta

$$H^e = \frac{k^e}{4} \begin{bmatrix} b_1^2 / \Delta^e & -2 \cot \varphi & -2 \cot \delta \\ -2 \cot \varphi & c_1^2 / \Delta^e & -2 \cot \vartheta \\ -2 \cot \delta & -2 \cot \vartheta & a_1^2 / \Delta^e \end{bmatrix}$$

$$p^e = \frac{\Delta^e}{12} S^e \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$