ENEL

Lezione del 10.12.91

Sistema differenziale lineare del 1° ordine:

$$\boxed{Ha + P\frac{da}{dt} + q = 0}$$

Metodo spettrale

$$\boxed{Hu = \lambda P u} \quad (*)$$

$$a = U w(t) = w_1(t)u_1 + w_2(t)u_2 + \dots + w_n(t)u_n$$

$$HUw + PU\frac{dw}{dt} + q = 0$$

$$U^T H U w + U^T P U \frac{dw}{dt} + U^T q = 0$$

Problema agli autovalori

$$HU = PU\Lambda$$

$$\boxed{U^T P U = I}$$ autovettori P-ortogonali

$$U^T H U = \overbrace{U^T P U}^{I}\Lambda = \Lambda$$

$$\Lambda w + \frac{dw}{dt} + U^T q = 0$$

Posto $U^T q = -g$

$$\boxed{w_i(t) = w_i(0)\, e^{-\lambda_i t} + \frac{g_i}{\lambda_i}\left(1 - e^{-\lambda_i t}\right)} \quad (**)$$

$g = \text{cost}$

(*) $u_i^T P u_j = 0 \qquad u_i^T P u_i = 1$

(**) $\lambda_i w_i + \dot{w}_i - g_i = 0 \rightarrow$ omogenea ass./eq. caratteristica $\lambda_i + \alpha = 0$

$\alpha = -\lambda_i \rightarrow w_i^*(t) = \exp(-\lambda_i t)$; $\bar{w} = C$ sostituendo: $C = +g_i/\lambda_i$ 16-7

Sol. generale: $w_i(t) = A_i \exp(-\lambda_i t) + g_i/\lambda_i \rightarrow w_i(0) = A_i + g_i/\lambda_i$

Forward difference

$$[H]_t \{a\}_t + [P]_t \frac{\{a\}_{t+\Delta t} - \{a\}_t}{\Delta t} + \{q\}_t = 0$$

$$\varepsilon = \frac{1}{2} P_t \left.\frac{\partial^2 a}{\partial t^2}\right]_t \Delta t$$

Backward difference

$$[H]_{t+\Delta t} \{a\}_{t+\Delta t} + [P]_{t+\Delta t} \frac{\{a\}_{t+\Delta t} - \{a\}_t}{\Delta t} + \{q\}_{t+\Delta t} = 0$$

$$\varepsilon = \frac{1}{2} P_{t+\Delta t} \left.\frac{\partial^2 a}{\partial t^2}\right]_{t+\Delta t} \Delta t$$

Crank-Nicolson a)

$$[H]_{t+\Delta t/2} \{a\}_{t+\Delta t/2} + [P]_{t+\Delta t/2} \frac{\{a\}_{t+\Delta t} - \{a\}_t}{\Delta t} +$$

$$+ \{q\}_{t+\Delta t/2} = 0$$

$$\{a\}_{t+\Delta t/2} = \frac{1}{2}\left(\{a\}_{t+\Delta t} + \{a\}_t\right)$$

Sostituendo si ottiene:

$$\left(\frac{1}{2}[H]_{t+\Delta t/2} + \frac{[P]_{t+\Delta t/2}}{\Delta t}\right)\{a\}_{t+\Delta t} +$$

$$+\left(\frac{1}{2}[H]_{t+\Delta t/2} - \frac{[P]_{t+\Delta t/2}}{\Delta t}\right)\{a\}_t + \{q\}_{t+\Delta t/2} = 0$$

Crank-Nicolson b):

$$\left([H]_{t+\Delta t} + \frac{[P]_t + [P]_{t+\Delta t}}{\Delta t}\right)\{a\}_{t+\Delta t} + \left([H]_t - \frac{[P]_t + [P]_{t+\Delta t}}{\Delta t}\right)\{a\}_t +$$

$$+ \{q_t\} + \{q\}_{t+\Delta t} = 0$$

$$\varepsilon_a) = -\frac{1}{8}\left\{\frac{1}{3} P_t \left.\frac{\partial^3 a}{\partial t^3}\right]_t - H_t \left.\frac{\partial^2 a}{\partial t^2}\right]_t\right\}\Delta t^2$$

$$\varepsilon_b) = \frac{1}{4}\left\{\frac{1}{3} P_t \frac{\partial^3 a}{\partial t^3} + \left.\frac{\partial P}{\partial t}\right]_t \frac{\partial^2 a}{\partial t^2}\right\}\Delta t^2$$

Wilson - Clough:

Assunzione:

$$\{a\}_{t+\Delta t} = \{a\}_t + \frac{1}{2}\Delta t\left(\left\{\frac{\partial a}{\partial t}\right\}_{t+\Delta t} + \left\{\frac{\partial a}{\partial t}\right\}_t\right)$$

Dall'eq. differenziale si ha:

$$\left\{\frac{\partial a}{\partial t}\right\}_t = [P_t]^{-1}\left(-[H]_t\{a\}_t - \{p\}_t\right)$$

$$\left\{\frac{\partial a}{\partial t}\right\}_{t+\Delta t} = [P]^{-1}_{t+\Delta t}\left(-[H]_{t+\Delta t}\{a\}_{t+\Delta t} - \{p\}_{t+\Delta t}\right)$$

Sostituendo:

$$\left([H]_{t+\Delta t} + \frac{2[P]_{t+\Delta t}}{\Delta t}\right)\{a\}_{t+\Delta t} - \left([P]_{t+\Delta t}\cdot[P]^{-1}_t[H]_t + \right.$$

$$\left. + \frac{2[P]_{t+\Delta t}}{\Delta t}\right)\{a\}_t + [P]_{t+\Delta t}\cdot[P]^{-1}_t\{p\}_t + \{p\}_{t+\Delta t} = 0$$

$$\varepsilon = \frac{1}{12}P_{t+\Delta t}\left.\frac{\partial^3 a}{\partial t^3}\right|_t \Delta t^2$$

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$[H]$ e $[P]$ time-independent:

$$\left([H] + \frac{2}{\Delta t}[P]\right)\{a\}_{t+\Delta t} =$$

$$= \left(\frac{2}{\Delta t}[P] - [H]\right)\{a\}_t - \{q\}_t - \{q\}_{t+\Delta t}$$

$$\varepsilon = \frac{1}{12}\, P \left.\frac{\partial^3 a}{\partial t^3}\right|_t \Delta t^2$$

$$H a_t + P \frac{a_{t+\Delta t} - a_t}{\Delta t} + l_t$$

$$H a_{t+\Delta t} + P \frac{a_{t+\Delta t} - a_t}{\Delta t} + l_{t+\Delta t}$$

$\cancel{Ha}$

$$H \frac{\left(a_t + a_{t+\Delta t}\right)}{2} + P \frac{a_{t+\Delta t} - a_t}{\Delta t} + \frac{l_t + l_{t+\Delta t}}{2}$$

$$\left(\frac{H}{2} + \frac{P}{\Delta t}\right) a_{t+\Delta t} + \left(\frac{H}{2} - \frac{P}{\Delta t}\right) a_t + \frac{l_t + l_{t+\Delta t}}{2}$$

$$\left(H + 2P/\Delta t\right) a_{t+\Delta t} + \left(H - \frac{2P}{\Delta t}\right) a_t + l_t + l_{t+\Delta t}$$

Stabilità degli schemi alle differenze

Eq. alle differenze per un sistema differenziale lineare:

$$\boxed{A x_{t+\Delta t} = B x_t + q_t}$$

La stessa equazione controlla la propagazione di un eventuale errore iniziale ε_0:

$$A \varepsilon_{t+\Delta t} = B \varepsilon_t \qquad \text{da cui:}$$

$$\varepsilon_{n\cdot\Delta t} = [A^{-1}B]^n \varepsilon_0 = E^n \varepsilon_0 \qquad \text{con } E = A^{-1}B$$

Per la stabilità occorre che sia $\lim_{n\to\infty} E^n = 0$, cioè $\rho(E) < 1$.

Assumiamo che A e B non dipendano da t.

Forward difference:

$$A = \frac{P}{\Delta t} \qquad B = \frac{P}{\Delta t} - H \ , \qquad E = \Delta t\, P^{-1}\left(\frac{P}{\Delta t} - H\right) = (I - \Delta t\, P^{-1}H)$$

Si ha stabilità solo se $\rho(I - \Delta t\, P^{-1}H) < 1$ cioè se

$$|1 - \Delta t\, \rho(P^{-1}H)| < 1$$

$$-1 + \Delta t\, \rho(P^{-1}H) < 1 \qquad \rho(P^{-1}H) < \frac{2}{\Delta t} \quad \text{ovvero} \quad \boxed{\Delta t < \frac{2}{\rho(P^{-1}H)}}$$

Notare che $P^{-1}H$, essendo il prodotto di 2 matrici p.d., ha autovalori reali >0

Backward difference:

$$A = H + \frac{P}{\Delta t} \ , \quad B = \frac{P}{\Delta t} \ , \quad E = \left(H + \frac{P}{\Delta t}\right)^{-1}\frac{P}{\Delta t} = (I + \Delta t\, P^{-1}H)^{-1}$$

$$\left[\left(\frac{P}{\Delta t}\right)^{-1}\left(H + \frac{P}{\Delta t}\right)\right]^{-1}$$

$$\rho(I + \Delta t\, P^{-1}H) = 1 + \Delta t\, \rho(P^{-1}H) > 1$$

e quindi $\rho(E) < 1$ sempre

Stabilità assicurata (nell'ipotesi che $\lambda(P^{-1}H) > 0$)

Crank-Nicolson: se H e P sono simmetriche p.d., incondizionatamente stabile