Formulazioni deboli

Integrale di Galerkin:

$$\int_R (Au_n - f)\,\xi_i\, dR = 0 \qquad i = 1,2,\dots,n$$

La 1ª identità di Green abbassa l'ordine delle derivate su u_n e l'alza su ξ_i. Si tratta dell'uso, anche ripetuto, dell'integrazioni per parti. Esempio con A operatore differenziale del 2° ordine:

$$\int_R (Au_n)\,\xi_i\, dR = \int_R u_n A^* \xi_i\, dR + \int_{\partial R} \left[G(u_n)F(\xi_i) - F(u_n)G^*(\xi_i)\right] d\Gamma$$

A^*: operatore aggiunto di A ($A^* = A$ se A è simmetrico)

F, G: operatori differenziali che scaturiscono naturalmente dalla integrazione per parti

Esempi di formulazioni deboli: $A = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

$$\int_R (Au)\,v\, dR = \int_R \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) v\, dR = -\int_R \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dR$$

$$+ \int_{\partial R} \frac{\partial u}{\partial n} v\, d\Gamma = \int_R \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) u\, dR + \int_{\partial R} \frac{\partial u}{\partial n} v\, d\Gamma - \int_{\partial R} u \frac{\partial v}{\partial n}\, d\Gamma$$

(2 applicazioni del lemma di Green)

In questo caso $A = A^*$ (perché Laplace è simmetrico)
Inoltre $F = 1$ e $G = G^* = \partial/\partial n$

Ad ogni integrazione per parti corrisponde una
formulazione debole

Se le condizioni al contorno sono trattate correttamente, quando $n \to \infty$, la soluzione debole $u_n \to u$, soluzione della PDE

Importanza ed utilità delle formulazioni deboli

Esempio con l'eq. bi-armonica:

$$\overset{4}{\Delta} u = f \quad \text{cioè}$$

$$\overset{2}{\Delta}(\overset{2}{\Delta} u) - f = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} - f = 0$$

Integrale i-esimo di Galerkin:

$$\int_R (\overset{4}{\Delta} u_n - f)\xi_i \, dR = 0$$

1^a applicazione della formulazione debole:

$$\int_R \overset{2}{\Delta}(\overset{2}{\Delta} u_n)\xi_i \, dR = -\int_R \left[\frac{\partial}{\partial x}(\overset{2}{\Delta} u_n)\frac{\partial \xi_i}{\partial x} + \frac{\partial}{\partial y}(\overset{2}{\Delta} u_n)\frac{\partial \xi_i}{\partial y}\right] dR$$

$$+ \int_{\partial R} \frac{\partial}{\partial n}(\overset{2}{\Delta} u_n)\xi_i \, d\Gamma$$

$$\Downarrow$$

$$\circledast \quad -\int_R \left(\frac{\partial^3 u_n}{\partial x^3}\frac{\partial \xi_i}{\partial x} + \frac{\partial^3 u_n}{\partial x^2 \partial y}\frac{\partial \xi_i}{\partial y} + \frac{\partial^3 u_n}{\partial x \partial y^2}\frac{\partial \xi_i}{\partial x} + \frac{\partial^3 u_n}{\partial y^3}\frac{\partial \xi_i}{\partial y}\right) dR =$$

$$= -\int_R \left(\frac{\partial^3 u_n}{\partial x^3} + \frac{\partial^3 u_n}{\partial x \partial y^2}\right)\frac{\partial \xi_i}{\partial x}\, dR - \int_R \left(\frac{\partial^3 u_n}{\partial x^2 \partial y} + \frac{\partial^3 u_n}{\partial y^3}\right)\frac{\partial \xi_i}{\partial y}\, dR =$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$= \underbrace{-\int_R \Delta^2\left(\frac{\partial u_n}{\partial x}\right)\frac{\partial \xi_i}{\partial x}\, dR}_{\text{lemma di Green}} \underbrace{- \int_R \Delta^2\left(\frac{\partial u_n}{\partial y}\right)\frac{\partial \xi_i}{\partial y}\, dR}_{\text{lemma di Green}} =$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$= \int_R \left[\frac{\partial}{\partial x}\left(\frac{\partial u_n}{\partial x}\right)\frac{\partial^2 \xi_i}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial u_n}{\partial x}\right)\frac{\partial^2 \xi_i}{\partial x \partial y}\right] dR + \int_R \left[\frac{\partial}{\partial x}\left(\frac{\partial u_n}{\partial y}\right)\frac{\partial^2 \xi_i}{\partial x \partial y} + \frac{\partial}{\partial y}\left(\frac{\partial u_n}{\partial y}\right)\frac{\partial^2 \xi_i}{\partial y^2}\right] dR$$

$$- \int_{\partial R} \frac{\partial}{\partial n}\left(\frac{\partial u_n}{\partial x}\right)\frac{\partial \xi_i}{\partial x}\, d\Gamma - \int_{\partial R} \frac{\partial}{\partial n}\left(\frac{\partial u_n}{\partial y}\right)\frac{\partial \xi_i}{\partial y}\, d\Gamma =$$

$$= \int_R \left(\frac{\partial^2 u_n}{\partial x^2}\frac{\partial^2 \xi_i}{\partial x^2} + 2\frac{\partial^2 u_n}{\partial x \partial y}\frac{\partial^2 \xi_i}{\partial x \partial y} + \frac{\partial^2 u_n}{\partial y^2}\frac{\partial^2 \xi_i}{\partial y^2}\right) dR$$

$$- \int_{\partial R} \left[\frac{\partial}{\partial n}\left(\frac{\partial u_n}{\partial x}\right)\frac{\partial \xi_i}{\partial x} + \frac{\partial}{\partial n}\left(\frac{\partial u_n}{\partial y}\right)\frac{\partial \xi_i}{\partial y}\right] d\Gamma$$

$$-\int_R \left(\frac{\partial^3 u_n}{\partial x^3}\frac{\partial \xi_i}{\partial x} + \frac{\partial^3 u_n}{\partial x^2 \partial y}\frac{\partial \xi_i}{\partial y} + \frac{\partial^3 u_n}{\partial x \partial y^2}\frac{\partial \xi_i}{\partial x} + \frac{\partial^3 u_n}{\partial y^3}\frac{\partial \xi_i}{\partial y}\right) dR$$

$$+ \int_{\partial R} \frac{\partial}{\partial n}\left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2}\right)\xi_i \, d\Gamma = \int_R f \xi_i \, dR$$

vedi 4-3b → (*)

2ª applicazione :

$$\int_R \left(\frac{\partial^2 u_n}{\partial x^2}\frac{\partial^2 \xi_i}{\partial x^2} + 2\frac{\partial^2 u_n}{\partial x \partial y}\frac{\partial^2 \xi_i}{\partial x \partial y} + \frac{\partial^2 u_n}{\partial y^2}\frac{\partial^2 \xi_i}{\partial y^2}\right) dR \; -$$

$$-\int_{\partial R}\left[\frac{\partial}{\partial n}\left(\frac{\partial u_n}{\partial x}\right)\frac{\partial \xi_i}{\partial x} + \frac{\partial}{\partial n}\left(\frac{\partial u_n}{\partial y}\right)\frac{\partial \xi_i}{\partial y}\right] d\Gamma \; +$$

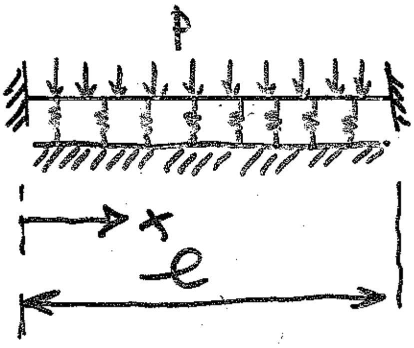
$$+ \int_{\partial R} \frac{\partial}{\partial n}\left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2}\right)\xi_i \, d\Gamma = \int_R f \xi_i \, dR$$

Integrali di linea = b.c. dell'eq. bi-armonica

Risultato importante : ξ_i possono ora appartenere ad uno spazio $W_2^{(2)}$ anzichè $W_2^{(4)}$!

Altro esempio: trave prismatica appoggiata su fondazione elastica :

$$EI \frac{d^4 u}{dx^4} + ku = p$$



$$u(0) = u(l) = \left.\frac{du}{dx}\right|_{x=0} = 0$$

Integrale variazionale di Galerkin:

$$\int_0^\ell \left(EI \frac{d^4 u_n}{dx^4} + k u_n - p \right) \xi_i \, dx = 0$$

Integrando per parti:

$$-\int_0^\ell EI \frac{d^3 u_n}{dx^3} \frac{d\xi_i}{dx} dx + \int_0^\ell (k u_n - p) \xi_i \, dx + EI \frac{d^3 u_n}{dx^3} \xi_i \Big|_0^\ell = 0$$

Integrando ancora:

$$\int_0^\ell EI \frac{d^2 u_n}{dx^2} \frac{d^2 \xi_i}{dx^2} dx + \int_0^\ell (k u_n - p) \xi_i \, dx + \underbrace{\cancel{EI \frac{d^3 u_n}{dx^3} \xi_i \Big|_0^\ell}}_{=0} - \underbrace{\cancel{EI \frac{d^2 u_n}{dx^2} \frac{d\xi_i}{dx} \Big|_0^\ell}}_{=0} = 0$$

Resta:

$$\int_0^\ell EI \frac{d^2 u_n}{dx^2} \frac{d^2 \xi_i}{dx^2} dx + \int_0^\ell k u_n \xi_i \, dx = \int_0^\ell p \xi_i \, dx$$

Le funzioni base ξ_i possono appartenere a $W^2_{(2)}$.