Equazioni differenziali alle derivate parziali (PDE)

2-D :

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}, \dots\right) = 0$$

Ω : dominio della PDE

Ordine di una PDE :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(Laplace)}$$

II° $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \varepsilon\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{(Burger)}$$

I° $$\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = 0 \qquad \text{(trasporto)}$$

PDE lineari :

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + e(x,y)\frac{\partial u}{\partial x} +$$
$$+ f(x,y)\frac{\partial u}{\partial y} + g(x,y)u + h(x,y) = 0$$

PDE quasi lineari (2-D) :

$$a\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial x^2} + b\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial x \partial y} + c\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\frac{\partial^2 u}{\partial y^2} +$$

PDE debolmente ("mildly") non lineari:

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+b(x,y)\frac{\partial^2 u}{\partial x\partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}+e(x,y)\frac{\partial u}{\partial x}+f(x,y)\frac{\partial u}{\partial y}+\overset{\Downarrow}{h(x,y,u)}=0$$

1) $\left[1+\left(\frac{\partial u}{\partial y}\right)^2\right]\frac{\partial^2 u}{\partial x^2}-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+\left[1+\left(\frac{\partial u}{\partial x}\right)^2\right]\frac{\partial^2 u}{\partial y^2}=0$ quasi lineare

2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\exp(u)=0$ "mildly" non lineare

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PDE 2-D della forma:

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}+e=0 \qquad (1)$$

$$p=\frac{\partial u}{\partial x}\quad q=\frac{\partial u}{\partial y}\quad r=\frac{\partial^2 u}{\partial x^2}\quad s=\frac{\partial^2 u}{\partial x\partial y}\quad t=\frac{\partial^2 u}{\partial y^2}$$

$\gamma \to x(\sigma),\ y(\sigma)$. Assumiamo che u, p, q, r, s et soddisfino (1)

$$\frac{dp}{d\sigma}=\frac{\partial p}{\partial x}\frac{dx}{d\sigma}+\frac{\partial p}{\partial y}\frac{dy}{d\sigma}=r\frac{dx}{d\sigma}+s\frac{dy}{d\sigma} \qquad (2)$$

$$\frac{dq}{d\sigma}=\frac{\partial q}{\partial x}\frac{dx}{d\sigma}+\frac{\partial q}{\partial y}\frac{dy}{d\sigma}=s\frac{dx}{d\sigma}+t\frac{dy}{d\sigma} \qquad (3)$$

Nella nuova notazione la (1) si scrive:

$$ar+bs+ct+e=0 \qquad (4)$$

Ricaviamo r dalla (2):

$$r = \frac{\dfrac{dp}{d\sigma} - s\dfrac{dy}{d\sigma}}{\dfrac{dx}{d\sigma}}$$

e t dalla (3): $$t = \frac{\dfrac{dq}{d\sigma} - s\dfrac{dx}{d\sigma}}{dy/d\sigma}$$

e sostituiamo in (4):

$$a\frac{\dfrac{dp}{d\sigma} - s\dfrac{dy}{d\sigma}}{\dfrac{dx}{d\sigma}} + bs + c\frac{\dfrac{dq}{d\sigma} - s\dfrac{dx}{d\sigma}}{dy/d\sigma} + e = 0 \Rightarrow$$

$$a\frac{d\sigma}{dx}\left(\frac{dp}{d\sigma} - s\frac{dy}{d\sigma}\right) + bs + c\left(\frac{dq}{d\sigma} - s\frac{dx}{d\sigma}\right)\frac{d\sigma}{dy} + e = 0$$

$$a\left(\frac{dp}{dx} - s\frac{dy}{dx}\right) + bs + c\left(\frac{dq}{dy} - s\frac{dx}{dy}\right) + e = 0$$

Moltiplichiamo l'equazione per dy/dx:

$$a\left[\frac{dp}{dx}\frac{dy}{dx} - s\left(\frac{dy}{dx}\right)^2\right] + bs\frac{dy}{dx} + c\left(\frac{dq}{\cancel{dy}}\frac{\cancel{dy}}{dx} - s\right) + e\frac{dy}{dx} =$$

$$s\left[a\left(\frac{dy}{dx}\right)^2 - b\frac{dy}{dx} + c\right] - \left(a\frac{dp}{dx}\frac{dy}{dx} + c\frac{dq}{dx} + e\frac{dy}{dx}\right) = 0$$

Scegliamo $\gamma(\sigma)$ tale che:

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0 \;\; \rightarrow \;\; \boxed{\frac{dy}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a}}$$

Su γ la PDE diventa:

$$a\frac{dp}{dx}\frac{dy}{dx} + c\frac{dq}{dx} + e\frac{dy}{dx} = 0 \quad \text{ODE in } p \text{ e } q$$

Consideriamo la conica:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

$b^2 - 4ac < 0$ ellisse → PDE ellittica

$b^2 - 4ac = 0$ parabola → PDE parabolica

$b^2 - 4ac > 0$ iperbole → PDE iperbolica

Se $b^2-4ac \geq 0$ si hanno le "linee caratteristiche" reali dove la PDE si trasforma in ODE.

Esempi:

1) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ $\quad a=c=1 \quad b=0 \quad b^2-4ac<0$

2) $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} = 0$ $\quad a=1 \quad b=c=0 \quad b^2-4ac=0$

3) $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0$ $\quad a=1 \quad b=0 \quad c=-1 \quad b^2-4ac>0$

Assumiamo p, q ed u assegnate su γ. Se la PDE vale su γ dobbiamo poter ricavare r, s e t. Si ottiene il sistema:

$$\begin{cases} r\dfrac{dx}{d\sigma} + s\dfrac{dy}{d\sigma} = \dfrac{dp}{d\sigma} \\ s\dfrac{dx}{d\sigma} + t\dfrac{dy}{d\sigma} = \dfrac{dq}{d\sigma} \\ ra + sb + tc = -e \end{cases}$$

$$\begin{vmatrix} \dfrac{dx}{d\sigma} & \dfrac{dy}{d\sigma} & 0 \\ 0 & \dfrac{dx}{d\sigma} & \dfrac{dy}{d\sigma} \\ a & b & c \end{vmatrix} \neq 0$$

sviluppo determinante

$$\frac{dx}{d\sigma}\left(c\frac{dx}{d\sigma} - b\frac{dy}{d\sigma}\right) + \frac{dy}{d\sigma}\left(a\frac{dy}{d\sigma}\right) = 0$$

Moltiplichiamo l'eq. per $\left(\dfrac{d\sigma}{dx}\right)^2$:

$$\Downarrow$$

$$a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$$

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

Le "linee caratteristiche" sono reali solo per PDE iperboliche e paraboliche

Condizioni iniziali ed al contorno per PDE del 2° ordine

1) Boundary conditions

2) Initial conditions

- Boundary value problem
- Initial value problem
- Initial boundary value problem

B.C. e I.C. :

$$\alpha(x,y,t)\, u(x,y,t) + \beta(x,y,t) \frac{\partial u}{\partial n}(x,y,t) = \gamma(x,y,t)$$

$$(x,y,t) \in \partial\Omega$$

- Dirichlet (principali) 1 $\beta = 0$
- Neumann (naturali) 2 $\alpha = 0$
- Cauchy (miste) 3 α e $\beta \neq 0$
- omogenee $\gamma = 0$