Outline

- Hyperbolic Conservation Laws
- Classical and Weak solutions
- The Riemann problem
- Finite Volume Methods in 1D
- Finite Volume Methods in 2D

The lectures about Finite Volume and Mixed Hybrid Finite Element will be on line at

http://dispense.dmsa.unipd.it/mazzia
Hyperbolic Conservation Laws

Finite Volumes (FV) schemes are most useful for modelling hyperbolic conservation laws, which, in their simplest (scalar) form may be expressed by the PDE

$$u_t + \nabla \cdot \vec{f} = 0$$

on a domain $\Omega$, with $u(x, t)$ specified on inflow boundaries.

Example: the equation for conservation of mass in groundwater contamination problem. Assumptions: no diffusion, no mechanical dispersion, density of the contaminant and the water constant, velocity of the fluid mixture constant.

Let $\rho(x, t)$ be the density of the contaminant at point $x$ and time $t$ and $v(x, t)$ be the velocity of the fluid mixture.
The total pollutant mass in two sections 1 and 2 is given by \( \int_{x_1}^{x_2} \rho(x, t) \, dx \). If no source or sinks are present, the mass can change only because of the mixture flowing across the endsections in \( x_1 \) or \( x_2 \).

The rate of mass change is given by:

\[
\frac{d}{dt} \int_{x_1}^{x_2} \rho(x, t) \, dx = \rho(x_1, t)v(x_1, t) - \rho(x_2, t)v(x_2, t)
\]

By integrating in time from \( t^k \) to \( t^{k+1} \) we obtain the integral form of the conservation law. Assuming \( \rho \) and \( v \) differentiable, with simple calculations we obtain the differential form of the conservation law:

\[
\rho_t + (\rho v)_x = 0
\]
Solution of hyperbolic conservation laws may be obtained if initial and boundary conditions are given. The simplest problem is the initial value problem, or \textit{Cauchy problem}, defined for $-\infty < x < \infty$ and $t \geq 0$. We must specify initial conditions only:

$$u_t + \nabla \cdot \vec{f} = 0$$

$$u(x, 0) = u_0(x) \quad -\infty < x < \infty.$$ 

Classical solutions of this problem are constant along the characteristics curves defined by $x(t)$ such that

$$\begin{cases} 
\frac{dx}{dt} = f'(u(x(t), t)) & t \geq 0 \\
 x(0) = x_0 
\end{cases}$$
Shock waves

$u$ is constant along these characteristics and the characteristics travel at constant velocity which is equal to $f'(u_0(x_0))$.

Generally, classical solutions exist only for $t \in [0, T^*)$ where

$$T^* = -\frac{1}{\inf_x u'_0(x)f''(u_0(x))}.$$ 

At the time $t = T^*$ the characteristics first cross, the function $u(x, t)$ has an infinite slope – the wave is said to break by analogy with waves on a beach – and a shock forms.
Weak solutions

To allow discontinuities, which arise in a natural way in this situation, we define a weak solution of conservation law.

**Definition (Weak solution)**

A function \( u(x, t) \), bounded and measurable, is called a weak solution of the conservation law, if for each \( \phi \in C^\infty(\mathbb{R} \times \mathbb{R}^+) \), the following equality holds:

\[
\int_0^{\infty} \int_{-\infty}^{+\infty} \left[ \phi_t u + \phi_x f(u) \right] \, dx \, dt = - \int_{-\infty}^{+\infty} \phi(x, 0) u(x, 0) \, dx.
\]
Definition (The Riemann problem)

A Riemann problem is the conservation law together with initial data consisting of two constant states separated by a single discontinuity,

$$u_0(x) = \begin{cases} u_l & x < 0, \\ u_r & x > 0. \end{cases}$$

Example: The Riemann problem in Burgers’ equation

$$u_t + \left( \frac{1}{2} u^2 \right)_x = 0$$
If $u_l > u_r$, there is a unique weak solution,

$$u(x, t) = \begin{cases} 
    u_l & x < s t \\
    u_r & x > s t.
\end{cases}$$

$$s = \frac{(u_l + u_r)}{2}$$

is the shock speed.

If $u_l < u_r$, there are infinitely many weak solutions, since between the points $u_l t < x < u_r t$, there is no information available from the characteristics. To determine the correct physical behavior we adopt the vanishing viscosity approach by adding the term $\epsilon u_{xx}$ depending on small parameter $\epsilon$

$$u_t + \left( \frac{1}{2} u^2 \right)_x = \epsilon u_{xx}. $$
If the initial data is smooth and $\epsilon$ very small then, before the wave begins to break, the $\epsilon u_{xx}$ term is negligible compared to other terms and the solutions to the two PDEs look nearly identical. As the wave begins to break, the term $u_{xx}$ grows much faster than $u_x$ and at some point the $\epsilon u_{xx}$ term is comparable to the other terms and begins to play a role. This term keeps the solution smooth for all time, preventing the breakdown of solutions that occurs for the hyperbolic problem. The weak solution is called rarefaction wave or expansion fan:

$$u(x, t) = \begin{cases} 
  u_l & x < u_l t \\
  x/t & u_l t \leq x \leq u_r t \\
  u_r & x > u_r t 
\end{cases}$$
Shock wave

Rarefaction wave
Some common numerical schemes

To solve numerically hyperbolic conservation laws some common numerical schemes are:

- Finite Differences
- Finite Volumes
- Finite Elements
- Discontinuous Galerkin
- ... 

We will see more in detail (but not too much...) the Finite Volume approach.
Properties of the ideal scheme

The ideal scheme for the resolution of hyperbolic conservation laws would reflect the properties of the underlying physical and mathematical system:

- **Conservative**: for correct capturing of discontinuities;
- **Accurate**: obvious....;
- **Free of spurious oscillations**: spurious oscillations have no physical meaning. This goal is remarkably difficult to achieve;
- **Continuous**: for convergence to the steady state solution.
In the Finite Volume method the three main steps to follow are:

- Partition the computational domain into **control volumes** (or **control cells**) - which are not necessarily the cells of the mesh.

- Discretize the **integral formulation** of the conservation laws over each control volume (by applying the divergence theorem).

- Solve the resulting set of algebraic equations or update the values of the dependent variables.
We will see two FV schemes in 1D: Godunov’s method and van Leer’s method.

- Godunov’s method uses the exact solution of the local Riemann problem and does not produce oscillations around discontinuities, but it is only first order accurate and the solutions display large numerical viscosity.

- van Leer’s method can be viewed as a generalization of Godunov’s method, it is a high resolution method, characterized by second order accuracy on smooth solutions and the absence of spurious oscillations.

Both methods are conservative.
**Conservative method**

**Definition**

Given a uniform grid with time step $\Delta t$ and spatial mesh size $\Delta x$, a numerical method is said to be conservative if the corresponding scheme can be written as:

$$v_{j}^{n+1} = v_{j}^{n} - \lambda (g_{j+\frac{1}{2}}^{n} - g_{j-\frac{1}{2}}^{n}), \quad j \in \mathbb{Z} \quad n \geq 0$$

where $v_{j}^{n}$ approximates $u(x_{j}, t^{n})$ at the point $(x_{j} = j\Delta x, t^{n} = n\Delta t)$, $\lambda = \frac{\Delta t}{\Delta x}$ and $g : \mathbb{R}^{2k} \rightarrow \mathbb{R}$ is a continuous function, called the *numerical flux* (function), that defines a $(2k + 1)$-point scheme.

$$g_{j+\frac{1}{2}}^{n} = g(v_{j-k+1}^{n}, \ldots, v_{j+k}^{n}).$$
Motivation of the conservative scheme

- $v_j^n$ can be view as an approximation of

$$\bar{u}_j^n = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n) \, dx.$$  

$\bar{u}_j^n$ is the average of $u(\cdot, t^n)$ on the cell $[x_{j-1/2}, x_{j+1/2}]$ (where $x_{j\pm1/2} = x_j \pm \frac{\Delta x}{2}$).

- From the integral form of the conservation law, we have:

$$\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^{n+1}) \, dx = \int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n) \, dx$$

$$- \left[ \int_{t^n}^{t^{n+1}} f(u(x_{j+1/2}, t)) \, dt - \int_{t^n}^{t^{n+1}} f(u(x_{j-1/2}, t)) \, dt \right].$$
Dividing by $\Delta x$ and using $\overline{u}_j^n$ we get

$$
\overline{u}_j^{n+1} = \overline{u}_j^n - \frac{1}{\Delta x} \left[ \int_{t_n}^{t_{n+1}} f(u(x_{j+\frac{1}{2}}, t)) \, dt - \int_{t_n}^{t_{n+1}} f(u(x_{j-\frac{1}{2}}, t)) \, dt \right].
$$

The numerical flux function can be considered as an average flux through $x_{j+\frac{1}{2}}$ over the time interval $[t_n, t_{n+1}]$,

$$
\overline{g}_{j+\frac{1}{2}}^n = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{j+\frac{1}{2}}, t)) \, dt.
$$
1. **Projection step** Given data $v_j^n$ at time $t^n$, construct a piecewise constant function $\hat{v}_j^n(x, t^n)$ defined by

$$\hat{v}_j^n(x, t^n) = v_j^n \quad x_{j-1/2} \leq x \leq x_{j+1/2}.$$
2. **Evolution step** Solve the local Riemann problem at the cell interfaces, that is, on each subinterval $[x_j, x_{j+1}]$ and for $t \geq t^n$, solve

\[
\begin{cases}
\frac{\partial \hat{v}_j^n}{\partial t} + \frac{\partial f(\hat{v}_j^n)}{\partial x} = 0 \\
\hat{v}_j^n(x, t^n) = \begin{cases} v_j^n, & x_j < x < x_{j+1}/2 \\
v_{j+1}^n, & x_{j+1}/2 < x < x_{j+1} \end{cases}
\end{cases}
\]

3. **Projection step** Define the approximation $v_j^{n+1}$ at time $t^{n+1}$ by averaging the Riemann problem solution $\hat{v}_j^n$ at the time $t^{n+1}$, so that

\[
v_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \hat{v}_j^n(x, t^{n+1}) \, dx.
\]

These values are then used to define new piecewise constant data $\hat{v}_j^{n+1}(x, t^{n+1})$ and the process repeats.
In the simple case of a linear advection equation \( u_t + au_x = 0 \), with \( a > 0 \), the third step gives

\[
v_j^{n+1} = v_j^n - a \frac{\Delta t}{\Delta x} (v_j^n - v_{j-1}^n).
\]
The CFL (Courant-Friedrichs-Lewy) condition

\[ \lambda \max_u |f'(u)| \leq \frac{1}{2}, \]

where \( \lambda = \frac{\Delta t}{\Delta x} \) is a necessary stability condition stating that the domain of dependence of the method includes the domain of dependence of the PDE.
van Leer’s method

- First and third steps of Godunov’s methods are of a numerical nature and can be modified without influencing the physics, in order to define spatially high order accurate scheme.
- van Leer’s scheme is a spatially second order accurate scheme (known also as MUSCL (Monotone Upstream-centered Scheme for Conservation Laws) method).
- The straightforward replacement of the first-order scheme by a second-order accurate interpolation leads to the generation of oscillations around discontinuities.
To overcome this limitations, non linear components are introduced.

This concept was introduced by van Leer under the form of *limiters*, i.e. functions that control the gradient of the computed solution to prevent the appearance of unphysical over/under shoots.
1 Reconstruction step: the dependent variable is interpolated using a piecewise linear function \( \hat{v} \) starting from the cell averages \( v^n_j, v^n_{j-1}, v^n_{j+1} \). To this end define \( S^n_j \) to be the slope on the \( j \)th cell calculated using \( v^n_{j-1} \) and \( v^n_{j+1} \). Then

\[
\hat{v}^n(x) = \begin{cases} 
  v^n_j + (x - x_j) \frac{S^n_j}{\Delta x} & x_{j-1/2} < x < x_{j+1/2}, \\
  \hat{v}^n(x_{j-1/2}) & x \leq x_{j-1/2} \\
  \hat{v}^n(x_{j+1/2}) & x \geq x_{j+1/2}.
\end{cases}
\]

Note that taking \( S^n_j = 0 \) for all \( j \) and \( n \) recovers Godunov’s method;
2 **Evolution step**: the waves are propagated across cell interfaces according to an exact or approximate solution of a local Riemann problem that uses the interpolated values $\hat{v}^n(x)$ as initial conditions. One solves

$$\begin{cases} \frac{\partial}{\partial t} w + \frac{\partial}{\partial x} f(w) = 0 & x \in \mathbb{R}, \ t^n \leq t \leq t^{n+1} \\ w(x, t^n) = \hat{v}^n(x). \end{cases}$$

This step yields $w(\cdot, t^{n+1})$.  

3 **Cell-averaging step**: $v_j^{n+1}$ is obtained by projecting the solution $w(x, t^{n+1})$ onto the piecewise constant functions

$$v_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} w(x, t^{n+1}) \, dx.$$
a) reconstruction; b) limiting; c) evolution; d) cell-averaging step.
**Slope limiter**

The numerical scheme is completely defined after specification of the slope limiter $S_j^n$. Poor choice of slopes may give oscillations in the solution. Generation of oscillations can be prevented by acting on their production mechanism and introducing nonlinear correction factors, so called the limiters, that force the method to be *total variation diminishing* (TVD).

**Definition**

A numerical method to solve hyperbolic conservation laws is called total variation diminishing if

\[
\sum_{j=-\infty}^{+\infty} |v_{j+1}^{n+1} - v_j^{n+1}| \leq \sum_{j=-\infty}^{+\infty} |v_{j+1}^n - v_j^n|
\]
To obtain a slope limiter, let

\[ S^n_j = \hat{S}^n_j \Phi^n_j \]

where \( \hat{S}^n_j \) represents the actual slope approximation and \( \Phi_j = \Phi(\theta^n_j) \) is a limiter function, defined in such a way that the method is TVD and \( \theta^n_j \) is the ratio of two consecutive gradients, i.e.:

\[ \theta^n_j = \frac{V^n_j - V^n_{j-1}}{V^n_{j+1} - V^n_{j}}. \]
Various limiter functions can be found in literature. For example:

- **van Leer’s limiter**:
  \[
  \Phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|},
  \]

- **minmod limiter** that represents the lowest boundary of the second-order TVD region:
  \[
  \Phi(\theta) = \begin{cases} 
  \min(\theta, 1) & \text{if } \theta > 0 \\
  0 & \text{if } \theta \leq 0.
  \end{cases}
  \]

- **superbee limiter**, that represents the upper limit of the second-order TVD region and has been introduced by Roe:
  \[
  \Phi(\theta) = \max[0, \min(2\theta, 1), \min(\theta, 2)].
  \]
Numerical example with periodic boundary conditions

\[ u_t + u_x = 0, \] with initial condition \( u_0(x) \) periodic of period 2 defined on the interval \([-1, 1]\) as

\[
\begin{align*}
    u_0(x) = & \begin{cases} 
    1 & -1 \leq x \leq -0.75 \\
    0 & -0.75 < x < -0.25 \\
    1 & -0.25 \leq x \leq 0.25 \\
    0 & 0.25 < x < 0.75 \\
    1 & 0.75 \leq x \leq 1
    \end{cases}
\end{align*}
\]

We solve this problem at time \( t = 2 \text{ s} \) with \( \lambda = 0.5 \), by setting \( \Delta t = 1 \times 10^{-2} \text{ s} \) and \( \Delta x = 2 \times 10^{-2} \text{ m} \).
(a) Godunov’s scheme

(b) van Leer’s scheme
van Leer limiter

(c) van Leer’s scheme
minmod limiter

(d) van Leer’s scheme
superbee limiter
Numerical example with non periodic boundary conditions

\[ u_t + u_x = 0, \] with the following initial condition:

\[ u(x, 0) = u_0(x) = \begin{cases} 
\sin(\pi x) & 0 \leq x \leq 2 \\
0 & \text{otherwise.} 
\end{cases} \]

We solve this problem at time \( t = 2 \) s with \( \lambda = 0.5 \) and \( \Delta x = 4 \times 10^{-2} \) m.
(a) Godunov’s scheme

(b) van Leer’s scheme
van Leer limiter

(c) van Leer’s scheme
minmod limiter

(d) van Leer’s scheme
superbee limiter