Mixed Hybrid Finite Element Method: an introduction
First lecture

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Some observation on the Finite Element method
The Mixed Finite Element method
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Numerical results
It is well known that the classical Finite Element (FE) method minimizes the so-called energy functional in a well chosen space of admissible functions $W$.

$$\inf_{w \in W} J(w).$$

If the functional $J(\cdot)$ is differentiable, the minimum (whenever it exists) will be characterized by a variational equation.

The FE method is based on a few simple ideas:

- The domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, in which the problem is posed, is partitioned into a set of simple subdomains, called elements.

- These elements may be triangles, quadrilaterals, tetrahedra.
A space $W$ of functions defined on $\Omega$ is then approximated by *simple* functions, defined on each element with suitable matching conditions at interfaces. Simple functions are commonly polynomials or functions obtained from polynomials by a change of variables.

A FE method can only be considered in relation with a variational principle and a functional space. Changing the variational principle and the space in which it is posed leads to a different FE approximation, even if the solution for the continuous problem can remain the same.
In the Mixed Finite Element (MFE) method two different Finite Element spaces are used. Let us consider the simple elliptic equation:

\[-\nabla \cdot (D \nabla c) = f \quad \text{in } \Omega\]
\[c = 0 \quad \text{on } \partial \Omega\]

where $D = D(x)$ is the dispersion tensor, and $c$ represents the concentration.

Introducing the dispersive flux $\tilde{G}$, we can write the Fick law,

\[\tilde{G} = -D \nabla c.\]

It could be desirable to approximate $\tilde{G}$ and $c$ simultaneously using a different Finite Element space for each variable.
With this purpose the elliptic problem is decomposed into a first order system as follows:

\[ \tilde{\mathbf{G}} + D \tilde{\nabla} c = 0 \quad \text{in } \Omega \]
\[ \tilde{\nabla} \cdot \tilde{\mathbf{G}} = f \quad \text{in } \Omega \]
\[ c = 0 \quad \text{on } \partial\Omega \]

The first of the above equations can be written as

\[ D^{-1} \tilde{\mathbf{G}} + \tilde{\nabla} c = 0 \quad \text{in } \Omega. \]
Multiplying by test functions and integrating by parts we obtain the following weak formulation

\[ \int_{\Omega} D^{-1} \vec{G} \cdot \vec{w} \, d\Delta - \int_{\Omega} c \vec{\nabla} \cdot \vec{w} \, d\Delta = 0 \quad \forall \vec{w} \in H(\text{div}, \Omega) \]

\[ \int_{\Omega} \psi \vec{\nabla} \cdot \vec{G} \, d\Delta = \int_{\Omega} f\psi \, d\Delta \quad \forall \psi \in L^2(\Omega) \]

where

\[ H(\text{div}, \Omega) = \{ \vec{w} \in L^2(\Omega)^d : \vec{\nabla} \cdot \vec{w} \in L^2(\Omega) \} \]

Hilbert space

The weak formulation involves the divergence of the solution and test functions and not arbitrary first derivatives. Thus we work with the space \( H(\text{div}, \Omega) \) formed by piecewise polynomial vector functions with continuous normal component.
In order to define finite element approximations to the solution \((\vec{G}, c)\), we need to have finite element subspaces of \(H(\text{div}, \Omega)\) and \(L^2(\Omega)\).

Let \(T = \{T_i\}_{i=1}^m\) be a triangulation of \(\Omega\), i.e. \(\Omega = \bigcup_{T_i \in T} T_i\) with diameter \(\leq h\). The triangulation is admissible if the intersection of two triangles is either empty, or a vertex, or a complete side.

Thus, we have to construct piecewise polynomials spaces \(W_h\) and \(\Psi_h\) associated with \(T_i\) such that \(W_h \subset H(\text{div}, \Omega)\) and \(\Psi_h \subset L^2(\Omega)\).
The MFE approximation \((\tilde{G}_h, c_h) \in W_h \times \psi_h\) is defined by

\[
\begin{align*}
\int_{\Omega} D^{-1} \tilde{G}_h \cdot \vec{w} \, d\Delta - \int_{\Omega} c_h \nabla \cdot \vec{w} \, d\Delta &= 0 \quad \forall \vec{w} \in W_h \\
\int_{\Omega} \psi \nabla \cdot \tilde{G}_h \, d\Delta &= \int_{\Omega} f \psi \, d\Delta \quad \forall \psi \in \Psi_h.
\end{align*}
\]

In order to have stability and convergence \(W_h\) and \(\psi_h\) can not be chosen arbitrarily but they have to be related.

- We assume that
  \[
  \nabla \cdot W_h = \psi_h.
  \]

- Let \(\pi_2 \tilde{G}\) be the \(L^2\)-projection of \(\tilde{G}\) into \(\psi_h\). The following equality must hold:
  \[
  \int_{\Omega} \nabla \cdot (\tilde{G} - \pi_2 \tilde{G}) \psi \, d\Delta = 0 \quad \forall \tilde{G} \in H^1(\Omega)^d, \quad \forall \psi \in \Psi_h
  \]

where \(H^1(\Omega)\) is the Sobolev space \((H^1(\Omega) = \{v \in L^2(\Omega) : D^1 v \in L^2(\Omega) \quad D^1 v \text{ partial derivative}\})\).
We can state the following theorem:

**Theorem**

The problem of finding a pair of functions \((\vec{\mathcal{G}}, c) \in H(\text{div}, \Omega) \times L^2(\Omega)\) such that the weak form of the elliptic problem holds has a unique solution. In addition, \(c\) is the solution of the elliptic problem and \(\vec{\mathcal{G}} = -D\vec{\nabla}c\).

Proof is omitted.
Numerical implementation of the MFE method

A MFE scheme is given by defining the spaces $W_h$ and $\Psi_h$ approximating $W \subset H(\text{div}, \Omega)$ and $\Psi \subset L^2(\Omega)$ respectively. We use the Raviart-Thomas spaces defined on a generic element $T_l \subset \Omega$ as

$$RT_k = (P_k)^d + xP_k$$

where $k$ is an integer $\geq 0$ and $P_k$ is the space of polynomials of degree $\leq k$.

The dimension of $RT_k$ is given by

$$\dim RT_k = \begin{cases} (k + 1)(k + 3) & \text{for } d = 2 \\ \frac{1}{2}(k + 1)(k + 2)(k + 4) & \text{for } d = 3. \end{cases}$$
Lemma

For $\vec{w}_h \in RT_k$ the following relations hold:

$$\vec{\nabla} \cdot \vec{w}_h \in P_k$$
$$\vec{w}_h \cdot \vec{n}|_{\partial T_l} \in R_k.$$

where $R_k$ is the polynomial space defined on the edges $e_j$ of each element $T_l$:

$$R_k = \{ \phi : \phi \in L^2(\partial T_l), \phi|_{e_j} \in P_k, \forall e_j \}.$$
We consider the Raviart-Thomas space of degree zero, whereby the functions $\vec{G}$ and $c$ can be approximated by:

$$\vec{G} \simeq \tilde{\vec{G}} = \sum_{l=1}^{m} g_l \vec{w}_l$$

$$c \simeq \tilde{c} = \sum_{l=1}^{m} c_l \psi_l$$

where $\vec{w}_l$ and $\psi_l$ are vector and scalar basis functions. Since we are considering the $RT_0$ spaces, $\vec{w}_l$ are first order polynomial of the type:

$$\vec{w}_l = \begin{pmatrix} ax + b \\ ay + c \end{pmatrix},$$

while $\psi_l$ are $P_0$ polynomials equal to one on element $T_i$ and zero elsewhere.
Lemma

Given a triangle $T_l$ with edges $e_j$, $j = 1, 2, 3$, the following relationships hold for $\vec{w}_l \in W_h(T_l)$:

\[
\nabla \cdot \vec{w}_l \in P_0
\]
\[
\vec{w}_l \cdot \vec{n}_j \in R_0 \quad j = 1, 2, 3
\]

where $\vec{n}_j$ is the outward normal to edge $e_j$ of $T_l$ and $R_0$ is the polynomial space defined on edge $e_j$ of $T_l$ as

\[
R_0 = \{ \phi : \phi \in L^2(\partial T_l), \phi|_{e_j} \in P_0, \forall e_j \}.
\]
Proof.

The first relation is very simple to prove:

\[ \nabla \cdot \vec{w}_l = 2a \in P_0. \]

The second relation says that the restriction of \( \vec{w}_l \) to each of the edges of \( T_l \) coincides with a polynomial of degree zero, i.e. a constant. Let \( T_l \) be an element with nodes \( P_1 = (x_1, y_1), P_2 = (x_2, y_2), P_3 = (x_3, y_3) \). The normal to the edge \( e_1 = P_1P_2 \) is given by:

\[ \vec{n}_l^1 = \begin{pmatrix} y_2 - y_1 \\ x_1 - x_2 \end{pmatrix}. \]
Noting that the line passing through the points $P_1$ and $P_2$ has equation:

$$x(y_2 - y_1) + y(x_1 - x_2) = x_1(y_2 - y_1) + y_1(x_1 - x_2),$$

the inner product $\vec{w}_i \cdot \vec{n}_i^l$ can be written as

$$\vec{w}_i \cdot \vec{n}_i^l = (ax + b)(y_2 - y_1) + (ay + c)(x_1 - x_2) = a[x_1(y_2 - y_1) + y_1(x_1 - x_2)] + b(y_2 - y_1) + c(x_1 - x_2) = \text{const.}$$
On each element $T_i$ we choose the RT0 basis functions $\vec{w}_i^j$ ($i = 1, 2, 3$) of the following form:

$$
\vec{w}_i^j = \begin{pmatrix}
 a_i^j x + b_i^j \\
 a_i^j y + c_i^j
\end{pmatrix}
$$

It is satisfied the following property (Kronecker property):

$$
\int_{e_{ij}} \vec{w}_i^j \cdot \vec{n}_i^j \, dS = \delta_{ij} = \begin{cases} 
1 & \text{if } j = i \\
0 & \text{otherwise}
\end{cases}
$$

where $\vec{n}_i^j$ is the outward normal to $e_i$. 
We can now derive the coefficients $a_i^l$, $b_i^l$ and $c_i^l$ coefficients in the following way (for sake of simplicity, we will omit the sub- super- scripts $l$, $i$).

Given the triangle $T_l$ with edges $e_1, e_2, e_3$ and nodes $P_1 = (x_1, y_1), P_2 = (x_2, y_2), P_3 = (x_3, y_3)$, the normal to edge $e_1 = P_1P_2$ is derived by:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

that is $\alpha(x - x_1) + \beta(y - y_1) = 0$, where

$$\alpha = y_2 - y_1, \quad \beta = x_1 - x_2.$$ 

The normalized components of the normal to $e_1$ are then (with no consideration about inner or outward normal) :

$$n_x = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}, \quad n_y = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}.$$
When imposing Kronecker property, we obtain the following result:

\[
\int_{e_i} \vec{\omega}^j \cdot \vec{n}^l \, dS = \int_{e_i} (ax + b)n_x + (ay + c)n_y \, dS = \\
a \int_{x_1}^{x_2} \left( x_1n_x + y_1n_y \right) \frac{1}{|n_y|} \, dx + \int_{x_1}^{x_2} \left( bn_x + cn_y \right) \frac{1}{|n_y|} \, dx = \\
a(x_1n_x + y_1n_y)\frac{|x_1 - x_2|}{|n_y|} + (bn_x + cn_y)\frac{|x_1 - x_2|}{|n_y|} = \delta_{ij}
\]

Since \(\frac{|x_1 - x_2|}{|n_y|} = \) is equal to
\[
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\alpha^2 + \beta^2}
\]
we obtain:

\[
\int_{e_i} \vec{\omega}^j \cdot \vec{n}^l \, dS = a(x_1\alpha + y_1\beta) + (b\alpha + c\beta) = \delta_{ij}
\]
The coefficients $a$, $b$, $c$ are chosen by the 2D classical Galerkin functions, $N_i$, $i = 1, 2, 3$, whose gradients are normal to the edges $e_{i+1}$ for $i = 1, 2$ while the gradient of $N_3$ is normal to $e_1$. They can be written as:

$$N_i = \frac{a_i + b_i x + c_i y}{2|T_i|},$$

where $|T_i|$ is the area of the triangle $T_i$ that can be written as:

$$|T_i| = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}.$$
In particular, the coefficients are:

\[
\begin{align*}
    a_1 &= \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \\
    a_2 &= -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} \\
    a_3 &= \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\
    b_1 &= \begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} \\
    b_2 &= -\begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} \\
    b_3 &= \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} \\
    c_1 &= -\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} \\
    c_2 &= \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} \\
    c_3 &= -\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}
\end{align*}
\]
It is simple to see that, for example, for the edge $e_1$ the following relations hold:

$$\alpha x_1 + \beta y_2 = -a_3, \quad \alpha = b_3, \quad \beta = c_3$$

Thus

$$\int_{e_1} \vec{w}^j \cdot \vec{n}_1 \, dS = -a_3 \alpha + (b_3 b + c_3 c).$$

The unknowns are now the coefficients $a, b, c$ of $\vec{w}^j$. Setting $j = 1$, from Kronecker property and substituting in the previous equation, we get:

$$-a_3 \alpha + b_3 b + c_3 c = 1$$
$$-a_1 \alpha + b_1 b + c_1 c = 0$$
$$-a_2 \alpha + b_2 b + c_2 c = 0$$

Note that $\sum_{i=1}^{3} b_i = \sum_{i=1}^{3} c_i =0$ and that $\sum_{i=1}^{3} a_i = 2|T_i|$. 
The system gives the following relationship between $a$ and $|T_i|$

\[-2|T_i|a = 1 \Leftrightarrow a = -\frac{1}{2|T_i|}.\]

Application of divergence theorem to $\int_T \nabla \cdot \mathbf{w}^1 \, dx \, dy$ assures that the value of $a$ is the same along the three edges of $T_i$ and is positive and equal to $\frac{1}{2|T_i|}$. Indeed

\[\int_T \nabla \cdot \mathbf{w}^1 \, dx \, dy = \int_{\partial T} \mathbf{w}^1 \cdot \mathbf{n} \, dS = \sum_{i=1}^{3} \int_{e_i} \mathbf{w}^1 \cdot \mathbf{n}^i \, dS = 1\]

But,

\[\int_T \nabla \cdot \mathbf{w}^1 \, dx \, dy = \int_T \left( \frac{\partial \mathbf{w}^1}{\partial x} + \frac{\partial \mathbf{w}^1}{\partial y} \right) \, dS = \int_T 2a \, dS = 2a|T_i|\]
Again, considering the first equation of the previous system and the measure of $T_i$, we obtain for $\vec{w}^1$:

$$b^I_1 = \frac{-x_3}{2|T_i|}, \quad c^I_1 = \frac{-y_3}{2|T_i|}. \quad \text{(1)}$$

Similarly, the coefficients for $\vec{w}^2$ and $\vec{w}^3$ are:

$$b^I_j = \frac{-x_{j-1}}{2|T_i|}, \quad c^I_j = \frac{-y_{j-1}}{2|T_i|}, \quad j = 2, 3 \quad \text{(2)}$$

while $a^I_i$ does not change, $a^I_i = \frac{1}{2|T_i|}$. Observe that with this choice of basis for the $V_h$ space, the components $g_1$, $g_2$ and $g_3$ of $\vec{G}_i$ on the element $T_i$ are the edge fluxes that is the velocity (or flux) along each edge of $T_i$. 

A. Mazzia (DMMMSA) MHFE method a.a. 2007/2008 24 / 24