Esercizio sugli EF triangolari

$$\Omega(u) = \iint_R \frac{1}{2}\left\{T_x\left(\frac{\partial u}{\partial x}\right)^2 + T_y\left(\frac{\partial u}{\partial y}\right)^2\right\} dx\,dy + \iint_R f u \, dR.$$

Si minimizza $\Omega(u_n)$ con $u_n = \sum_{i=1}^{n} a_i \xi_i(x,y)$

$$\frac{\partial \Omega(u_n)}{\partial a_i} = \iint_R \left\{T_x\left(\frac{\partial u_n}{\partial x}\right)\frac{\partial \xi_i}{\partial x} + T_y\left(\frac{\partial u_n}{\partial y}\right)\frac{\partial \xi_i}{\partial y}\right\} dx\,dy +$$

$$+ \iint_R f\left(\frac{\partial u_n}{\partial a_i}\right) dx\,dy =$$

$$= \iint_R \left\{T_x\left(a_1\frac{\partial \xi_1}{\partial x} + a_2\frac{\partial \xi_2}{\partial x} + \dots + a_n\frac{\partial \xi_n}{\partial x}\right)\frac{\partial \xi_i}{\partial x}\right\} \dots$$

$$+ \iint_R \left\{T_y\left(a_1\frac{\partial \xi_1}{\partial y} + a_2\frac{\partial \xi_2}{\partial y} + \dots + a_n\frac{\partial \xi_n}{\partial y}\right)\frac{\partial \xi_i}{\partial y}\right\} dx\,dy +$$

$$+ \iint_R f \, \xi_i(x,y)\, dx\,dy =$$

Questa è l'equazione i-esima di Ritz-Galerkin

a cui si aggiunge il termine noto

$$\iint_R f \xi_i \, dx\,dy$$

se il problema è di evoluzione il principio variazionale ristretto è:

$$\Omega(u) = \iint_R \frac{1}{2}\left\{ T_x\left(\frac{\partial u}{\partial x}\right)^2 + T_y\left(\frac{\partial u}{\partial y}\right)^2 \right\} dx\,dy + \iint_R u\left[f + S\frac{\partial u}{\partial t}\right] dx\,dy$$

si pone $u_n(x,y,t) = \sum_{i=1}^{n} a_i(t)\,\xi_i(x,y)$

$$\frac{\partial u_n}{\partial t} = \sum_{i=1}^{n} \frac{\partial a_i}{\partial t}\,\xi_i(x,y)$$

Pertanto il contributo dell' $\iint_R S\,u_n \left[\frac{\partial u_n}{\partial t}\right] dx\,dy$ è:

$$\frac{\partial}{\partial a_i}\left[\iint_R S\,u_n\left[\frac{\partial u_n}{\partial t}\right] dx\,dy\right] =$$

$$= \iint_R S\,\xi_i(x,y)\left[\frac{\partial a_1}{\partial t}\xi_1 + \frac{\partial a_2}{\partial t}\xi_2 + \dots + \frac{\partial a_n}{\partial t}\xi_n\right] dx\,dy$$

Questo contributo si aggiunge all'eq. i-esima

Termine noto dell'eq. i-esima è sempre

$$\iint_R f\,\xi_i\,dx\,dy$$

$$[H]\{a\} + \{q\} = 0$$

Eq. di filtrazione:

$$h_{ij} = \int_R \left(T_x \frac{\partial \xi_i}{\partial x}\frac{\partial \xi_j}{\partial x} + T_y \frac{\partial \xi_i}{\partial y}\frac{\partial \xi_j}{\partial y} \right) dx\, dy$$

Problem: time-dependent:

$$u_n(x,t) = \sum_{i=1}^{n} a_i(t)\, \xi_i(x)$$

$$\frac{\partial u_n}{\partial t} = \sum_{i=1}^{n} \frac{\partial a_i}{\partial t}\, \xi_i(x)$$

$$[H]\{a\} + [P]\left\{\frac{\partial a}{\partial t}\right\} + \{q\} = 0$$ lineare

$$p_{ij} = \int S\, \xi_i \xi_j\, dx\, dy$$

$$q_i = \iint_R f\, \xi_i\, dx\, dy$$

Applicazioni

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial y}\right) = S\frac{\partial h}{\partial t} + f(x,y,t) \qquad \text{in } \Omega$$

Condizioni di Neumann su δR_2:

$$k_x \frac{\partial h}{\partial x} n_x + k_y \frac{\partial h}{\partial y} n_y = q$$

+ Dirichlet.

Funzionale associato:

$$\Omega(h) = \iint_R \left\{ \frac{1}{2}\left[k_x\left(\frac{\partial h}{\partial x}\right)^2 + k_y\left(\frac{\partial h}{\partial y}\right)^2\right] + \left(S\frac{\partial h}{\partial t} + f\right)h \right\} dx\,dy - \int_{\delta R_2} q h \, d\Gamma$$

Elementi finiti triangolari:

$$h^e = \xi_i^e(x,y)h_i(t) + \xi_j^e(x,y)h_j(t) + \xi_m^e(x,y)h_m(t)$$

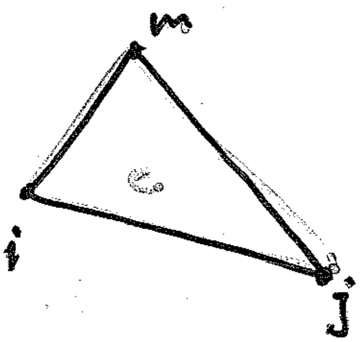
$$\xi_i^e(x,y) = (a_i + b_i x + c_i y)/2\Delta$$

$a_i = x_j y_m - x_m y_j$

$b_i = y_j - y_m$

$c_i = x_m - x_j$

Δ = area del triangolo ijm



$$\Omega(h) = \sum_e \Omega^e(h)$$

Contributo di e alle 3 derivate:

$$\frac{\partial \Omega^e}{\partial h_i} \qquad \frac{\partial \Omega^e}{\partial h_j} \qquad \frac{\partial \Omega^e}{\partial h_m}$$

$$\Omega^e = \iint_{\Delta_e} \left\{ \frac{1}{2}\left[k_x^e \left(\frac{\partial \xi_i^e}{\partial x} h_i + \frac{\partial \xi_j^e}{\partial x} h_j + \frac{\partial \xi_m^e}{\partial x} h_m \right)^2 + \right.\right.$$
$$\left. + k_y^e \left(\frac{\partial \xi_i^e}{\partial y} h_i + \frac{\partial \xi_j^e}{\partial y} h_j + \frac{\partial \xi_m^e}{\partial y} h_m \right)^2 \right] +$$
$$\left. + \left[S^e \left(\xi_i^e \frac{\partial h_i}{\partial t} + \xi_j^e \frac{\partial h_j}{\partial t} + \xi_m^e \frac{\partial h_m}{\partial t} \right) + f^e \right] (\xi_i^e h_i + \xi_j^e h_j + \xi_m^e h_m) \right\} dx\,dy$$

$$\frac{\partial \Omega^e}{\partial h_i} = \iint_{\Delta_e} \left\{ k_x^e \left(\frac{\partial \xi_i^e}{\partial x} h_i + \frac{\partial \xi_j^e}{\partial x} h_j + \frac{\partial \xi_m^e}{\partial x} h_m \right) \frac{\partial \xi_i^e}{\partial x} + \right.$$
$$+ k_y^e \left(\frac{\partial \xi_i^e}{\partial y} h_i + \frac{\partial \xi_j^e}{\partial y} h_j + \frac{\partial \xi_m^e}{\partial y} h_m \right) \frac{\partial \xi_i}{\partial y} +$$
$$\left. + \left[S^e \left(\xi_i^e \frac{\partial h_i}{\partial t} + \xi_j^e \frac{\partial h_j}{\partial t} + \xi_m^e \frac{\partial h_m}{\partial t} \right) + f^e \right] \xi_i^e \right\} dx\,dy$$

Esprimendo le derivate delle ξ_i:

$$\frac{\partial \Omega^e}{\partial h_i} = \iint_{\Delta_e} \frac{1}{2\Delta} \left\{ k_x^e (b_i h_i + b_j h_j + b_m h_m) \frac{b_i}{2\Delta} + \right.$$
$$\left. + k_y^e (c_i h_i + c_j h_j + c_m h_m) \frac{c_i}{2\Delta} \right\} dx\,dy +$$

$$+ \iint_{\Delta_e} \left[S^e \left(\xi_i^e \frac{\partial h_i}{\partial t} + \xi_j^e \frac{\partial h_j}{\partial t} + \xi_m^e \frac{\partial h_m}{\partial t} \right) + f^e \right] \xi_i^e \, dx\,dy$$

Matrice di rigidezza (locale):

$$H_{ik}^e = \frac{1}{4\Delta^e}\left(k_x^e b_i b_k + k_y^e c_i c_k\right) \qquad k = i, j, m$$

Matrice di capacità (locale):

$$P_{ij}^e = \iint_{\Delta_e} S^e \xi_i^e \xi_j^e \, dx\, dy = \frac{\Delta}{12} S^e \begin{cases} 2 & i = j \\ 1 & i \neq j \end{cases}$$

$$q_i = \iint_{\Delta_e} f^e \xi_i \, dx\, dy = \frac{\Delta}{3} f^e$$

Se i è nodo del contorno si ha:

$$\frac{\partial}{\partial h_i} \int_{\ell^e} q^e h^e \, d\Gamma = \int_{\ell^e} q^e \xi_i^e \, d\Gamma = q^e \frac{\ell^e}{2}$$

Infine si ottiene:

$$[H]\{h\} + [P]\left\{\frac{\partial h}{\partial t}\right\} + \{q\} = 0 \qquad \{h\} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{bmatrix}$$

Stazionario:

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial h}{\partial y}\right) = f(x, y)$$

$$\Downarrow$$

$$[H]\{h\} + \{q\} = 0$$

$[H]$ e $[P]$ sono matrici simmetriche d.p.

$$\Omega(h) = \frac{1}{2} \{h\}^T [H] \{h\} > 0$$

Equazione di ffusione - convezione:

$$\frac{\partial}{\partial x}\left(D_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_y \frac{\partial c}{\partial y}\right) - v_x \frac{\partial c}{\partial x} - v_y \frac{\partial c}{\partial y} = \frac{\partial c}{\partial t}$$

(con $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$ cioè campo di moto stazionario)

Applichiamo l'approccio variazionale di Galerkin:

$$H_{ij} = \iint_R \left\{ D_x \frac{\partial \xi_i}{\partial x}\frac{\partial \xi_j}{\partial x} + D_y \frac{\partial \xi_i}{\partial y}\frac{\partial \xi_j}{\partial y} \right\} dx\,dy$$

$$P_{ij} = \iint_R \xi_i \xi_j \, dx\,dy$$

$$B_{ij} = \iint_R \left(v_x \frac{\partial \xi_j}{\partial x} + v_y \frac{\partial \xi_j}{\partial y} \right) \xi_i \, dx\,dy$$

$$B_{ij} \neq B_{ji}$$

$$c_n = c_1 \xi_1(x,y) + c_2 \xi_2(x,y) + \dots + c_n \xi_n(x,y)$$

Se le velocità sono costanti nel triangolo si ha:

$$B_{ij}^e = \iint_{\Delta_e} \frac{1}{2\Delta} \left(v_x^e b_j + v_y^e c_j\right)\left(\frac{a_i + b_i x + c_i y}{2\Delta}\right) dx\,dy$$

$$= \frac{1}{6}\left(v_x^e b_j + v_y^e c_j\right)$$

$$B^e_{ij} \neq B^e_{ji}$$

Dal procedimento scaturisce l'equazione:

$$\left([H] + [B]\right)\{c\} + [P]\left\{\frac{\partial c}{\partial t}\right\} + \{q\} = 0$$

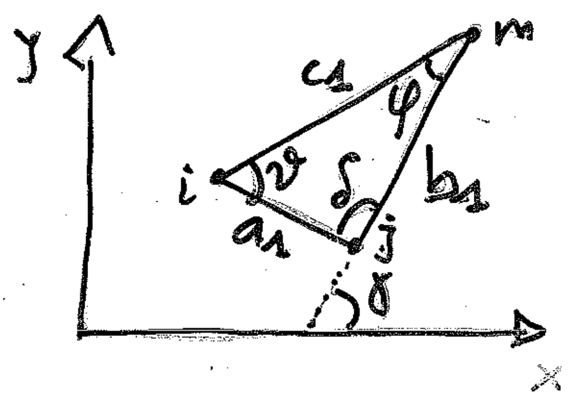
La matrice $[H]+[B]$ non è simmetrica ancorché sparsa. Il GCM in genere non converge. Si possono usare altri metodi del gradiente opportunamente precondizionato:

1- GMRES (Generalized Minimal Residual)
2- BiCGSTAB (Bi Conjugate Gradient Stabilized)
3- TFQMR (Transpose Free Quasi Minimal Residual)

Problema isotropo ($k_x = k_y$)

La matrice locale di rigidezza è:

$$H^e = \frac{k^e}{4\Delta^e}\begin{bmatrix} b_i^2 + c_i^2 & b_i b_j + c_i c_j & b_i b_m + c_i c_m \\ b_j b_i + c_j c_i & b_j^2 + c_j^2 & b_j b_m + c_j c_m \\ b_m b_i + c_m c_i & b_m b_j + c_m c_j & b_m^2 + c_m^2 \end{bmatrix}$$



$$\begin{array}{l|l} b_i = y_j - y_m = -b_1 \sin\gamma & c_i = x_m - x_j = b_1 \cos\gamma \\ b_j = y_m - y_i = -c_1 \sin(\gamma - \varphi) & c_j = x_i - x_m = -c_1 \cos(\gamma - \varphi) \\ b_m = y_i - y_j = a_1 \sin(\gamma + \delta) & c_m = x_j - x_i = -a_1 \cos(\gamma + \delta) \end{array}$$

Risulta

$$H^e = \frac{k^e}{4}\begin{bmatrix} b_1^2/\Delta^e & -2\cot\varphi & -2\cot\delta \\ -2\cot\varphi & c_1^2/\Delta^e & -2\cot\vartheta \\ -2\cot\delta & -2\cot\vartheta & a_1^2/\Delta^e \end{bmatrix}$$

$$p^e = \frac{\Delta^e}{12} S^e \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$